ELSEVIER

Journal of Pure and Applied Algebra 175 (2002) 141–185

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Pseudo-commutative monads and pseudo-closed 2-categories ☆

Martin Hyland[a,*], John Power[b]

[a] DPMMS, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK

[b] Laboratory for the Foundations of Computer Science, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, UK

Received 8 January 2001
Communicated by R. Street

Dedicated to Max Kelly on the occasion of his 70th birthday: a token of affection and respect

Abstract

Pseudo-commutative 2-monads and pseudo-closed 2-categories are defined. The former give rise to the latter: if T is pseudo-commutative, then the 2-category T-Alg, of strict T-algebras and pseudo-maps of algebras, is pseudo-closed. In particular, the 2-category of symmetric monoidal categories, is pseudo-closed. Subject to a biadjointness condition that is satisfied by T-Alg, pseudo-closed structure induces pseudo-monoidal structure on the 2-category. © 2002 Elsevier Science B.V. All rights reserved.

MSC: 18D05;18C15;18D15

1. Introduction

The theory of monads on a category provides an abstract syntax-free approach to universal algebra. Standard references are Mac Lane [15] and Barr and Wells [1]: Street [17] gives an illuminating abstract treatment. The basic ideas of monad theory have non-trivial analogues at the 2-categorical level. The theory of 2-monads as developed in [3] provides an abstract setting in which to study algebraic structure on 2-categories generally and the 2-category *Cat* in particular. The subject is unavoidably more subtle.

☆ This work has been done with the support of EPSRC grant GR/M56333 and a British Council grant.

* Corresponding author. Tel.: +44-1223-337-986.
E-mail addresses: m.hyland@dpmms.cam.ac.uk (M. Hyland), ajp@dcs.ed.ac.uk (J. Power).

PII: S0022-4049(02)00133-0

One has not just strict algebras but also pseudo-algebras though these will not concern us here; and even between strict algebras one has both strict and pseudo-maps of algebras and the important pseudo-maps give the subject its special edge.

In a similar fashion, the theory of commutative monads on a symmetric monoidal closed category $\mathscr{V}$ gives an abstract approach to algebra with commuting operations (generalised linear algebra). This theory is described in [11–13]. (Between them [11,13] show that a commutative monad is a symmetric monoidal monad: so the theory is covered by the abstract perspective of [17].) In the basic setting, $\mathscr{V}$ is complete and cocomplete and T is bounded, so that if T is commutative, the category T-Alg is itself symmetric monoidal closed. In this paper we describe an analogue of this theory at the 2-dimensional level.

One analogue is obvious. For a strictly commutative 2-monad T, say on Cat, not only is the category of strict T-algebras and strict maps symmetric monoidal closed, but so also is the corresponding 2-category. As Kelly [10] explains, that is just the enriched version of the theory given in [12]. This result is unsatisfactory for two reasons. In the first place while some 2-monads are commutative (for example, that for a category with terminal object), those of primary interest to us, notably that for small symmetric monoidal categories, are not commutative as the relevant hexagon

$$\begin{tikzcd}
TA \times TB \arrow[r, "t^*"] \arrow[d, "t"'] & T(TA \times B) \arrow[r, "T(t)"] & T^2(A \times B) \arrow[d, "\mu_{A\times B}"] \\
T(A \times TB) \arrow[r, "Tt^*"'] & T^2(A \times B) \arrow[r, "\mu_{A\times B}"'] & T(A \times B)
\end{tikzcd}$$

only commutes up to coherent isomorphism. Secondly for a genuinely higher dimensional algebra it is the 2-category T-Alg of strict T-algebras and pseudo-maps of T-algebras, as developed in [3], that is the focus of attention. Our goal is a study of appropriate monoidal and closed structure on the 2-category T-Alg. We are led to this by a desire to broaden the abstract setting for our study of wiring diagrams [9]. In addition we see this as an important step towards a general higher dimensional algebra.

Already in [10], Kelly adumbrated a theory of pseudo-commutativity for a 2-monad. His notion arose in the course of an investigation of a general notion of a pseudo-distributive law between two 2-monads: so his definition was designed so that a pseudo-commutativity for a 2-monad T on Cat gave rise to pseudo-distributivity between T and the 2-monad for small symmetric monoidal categories. Hence it allowed T to lift from Cat to the 2-category of small symmetric monoidal categories and strong monoidal functors. Thus, his result was a 2-categorical analogue of the folklore observation that a commutative monad on $Sets$ lifts to a monad on the category of commutative monoids. Here, we also present a notion of pseudo-commutativity for a 2-monad, but our main result is different from Kelly's, and we are led to a different formulation of the notion. Our notion of symmetric pseudo-commutativity is equivalent to the one sketched in [10], but we shall not prove that in detail here.

In mathematical experience, closed structure appears more canonical than monoidal structure: we understand the vector space of linear maps more readily than the tensor product of vector spaces. For commutative T, limits are used to give the closed structure (already in [12]) and colimits to give the monoidal structure on T-Alg. Again, limits in *Sets* seem easier than colimits; and more substantially limits in categories of algebras are created by the forgetful functor and so are easier than colimits. At the 2-dimensional level these intuitions become a technical distinction. As shown in [3], T-Alg has most of the flexible limits in the sense of [2] (all of them if T itself is flexible), while it only has bicolimits. Thus it is reasonable to focus on closed structure which should be stricter than monoidal structure.

We initially hoped that, if we had an isomorphism γ in the hexagon above satisfying suitable axioms, then T-Alg would be closed in the sense of Eilenberg and Kelly [7]; then we expected to use [3] to deduce that it was an example of what we would call a pseudo-monoidal closed 2-category. That almost worked, but one axiom for closedness from [7], namely that the map $A \to [I,A]$ be an isomorphism, cannot hold for non-trivial 2-monads: it implies that the categories of strict and of pseudo-maps from the free T-algebra $F1$ on 1 to an arbitrary T-algebra $\mathscr{A}$ are isomorphic, whereas in fact they are just equivalent. So to include serious examples we had to relax that axiom to the existence of an equivalence with good properties. With this relaxation and some reformulation, we can indeed place axioms on γ that allow us to deduce that T-Alg is closed. So we call this relaxed notion *pseudo-closedness*, and the main result of the paper is that a pseudo-commutativity for a 2-monad T on *Cat* makes T-Alg into a pseudo-closed 2-category. It follows that T-Alg is a pseudo-monoidal category, is closed as such, and the left adjoint from *Cat* to T-Alg is a strong pseudo-monoidal 2-functor, in that it sends finite products in *Cat* to the pseudo-monoidal structure of T-Alg up to coherent equivalence.

In view of the length of this paper we present a brief outline. We start in Section 2 by presenting a definition of pseudo-closed category based on [7]. The notion we develop is not the most general one could reasonably imagine, and that is deliberate. Just as closed categories arise from monoidal categories with an ordinary right adjoint, the most general notion should be obtained by considering a pseudo-monoidal bicategory (one object tricategory) in which $- \otimes A$ has a right biadjoint for each object A, and then axiomatising the resulting structure. However the closed structures we are interested in satisfy stricter conditions: that allows us to avoid cumbersome coherence concerns. Hence we define our notion of pseudo-closedness so that we have equality rather than coherent isomorphism wherever possible. One should regard this as a simple extension of the consideration in [3] of flexible limits rather than bilimits where possible. The other main ingredient of the paper is the notion of pseudo-commutativity. We give our axioms in Section 3, and run through some general calculations stemming from them, though our main interest is with symmetric pseudo-commutativities. We explain briefly why the notion of symmetric pseudo-commutativity is equivalent to the notion Kelly outlines in [10]. In Section 4 we present a little general background on the 2-category T-Alg as developed in [3], and give some elementary consequences for pseudo-commutative T. The pseudo-closed structure on T-Alg reflects features of multilinear algebra, so in Section 5 we describe 2-multicategorical structure on T-Alg. Finally, in Section 6 we define the pseudo-closed structure on T-Alg and show that it satisfies our axioms. We close with some remarks on the pseudo-monoidal structure induced by the pseudo-closed structure.

2. Pseudo-closed 2-categories

We present a notion of pseudo-closedness for a 2-category. Our notion is motivated by our main result, but we support it with an embedding theorem which generalises the situation for closed categories. Eventually, we shall want an analogue of the result of [7] which states (modulo details) that a closed category $\mathscr{V}$ with left adjoints to $[A,-]:\mathscr{V}\to\mathscr{V}$ is monoidal. Here we only give the briefest outline of a generalisation.

2.1. Pseudo-closedness: the definition

We give a definition of pseudo-closed 2-category following the spirit of Eilenberg and Kelly's definition of closed category [7].

Definition 1. A *pseudo-closed 2-category* consists of a 2-category $\mathscr{K}$ together with a 2-functor

$$[-,-]:\mathscr{K}^{\mathrm{op}}\times\mathscr{K}\to\mathscr{K}$$

and a 2-functor $V:\mathscr{K}\to Cat$, together with an object I of $\mathscr{K}$ and transformations, j,e,k, with components

- $j_A: I\to[A,A]$,
- $e_A:[I,A]\to A$ natural in A,
- $k_A=k_{A,B,C}:[B,C]\to[[A,B],[A,C]]$ natural in A,B and C,

with $V[-,-]=\mathscr{K}(-,-):\mathscr{K}^{\mathrm{op}}\times\mathscr{K}\to Cat$, and such that the following conditions (numbered as in [7]) hold:

1.
$$\begin{tikzcd} I \arrow[r, "j_B"] \arrow[dr, "j_{[A,B]}"'] & {[B,B]} \arrow[d, "k_A"] \\ & {[[A,B],[A,B]]} \end{tikzcd}$$

2.
$$\begin{tikzcd} {[A,C]} \arrow[r, "k_A"] \arrow[d, equal] & {[[A,A],[A,C]]} \arrow[d, "{[j_A,[A,C]]}"] \\ {[A,C]} & {[I,[A,C]]} \arrow[l, "e_{[A,C]}"] \end{tikzcd}$$

3.
$$\begin{tikzcd} {[C,D]} \arrow[r, "k_A"] \arrow[d, "k_B"'] & {[[A,C],[A,D]]} \arrow[r, "k_{[A,B]}"] & {[[[A,B],[A,C]],[[A,B],[A,D]]]} \arrow[d, "{[k_A,[[A,B],[A,D]]]}"] \\ {[[B,C],[B,D]]} \arrow[rr, "{[[B,C],k_A]}"'] & & {[[B,C],[[A,B],[A,D]]]} \end{tikzcd}$$

4.

$$\begin{tikzcd}
{[A,B]} \arrow[r, "k_I"] \arrow[dr, "{[e_A,B]}"'] & {[[I,A],[I,B]]} \arrow[d, "{[[I,A],e_B]}"] \\
& {[[I,A],B]}
\end{tikzcd}$$

5. The map

$$\mathscr{K}(I,[A,A]) = V[I,[A,A]] \to V[A,A] = \mathscr{K}(A,A)$$

induced by $e_{[A,A]}$ takes j_A to the identity 1_A.

In addition, we require that e be a well-behaved adjoint retract equivalence: we require transformations with components $i_A : A \to [I,A]$, and adjunctions $i_A \dashv e_A$ such that

- the unit $1_A \to e_A \cdot i_A$ is equal to the identity,
- the counit $i_A \cdot e_A \to 1_{[I,A]}$ is invertible, and
- the retraction $V(i_A \cdot e_A)$ takes $p : I \to A$ to the composite

$$I \xrightarrow{j_A} [A,A] \xrightarrow{[p,A]} [I,A] \xrightarrow{e_A} A.$$

We compare this definition with that of closed category in [7]. Of course given our aims, we ask now for 2-categories, 2-functors, and 2-natural transformations: moreover, as $\mathscr{K}(-,-)$ is a 2-functor into *Cat*, the codomain for the forgetful V should be *Cat*. Allowing for these changes, the definitions are parallel, in that our five conditions correspond to Eilenberg and Kelly's five axioms. The seemingly insignificant difference that Eilenberg and Kelly expressed their data and axioms in terms of an inverse of e rather than directly in terms of e is however substantial as we ask that e be a retract equivalence rather than an isomorphism. (We have no choice here if we are to include significant examples: the 2-category of small symmetric monoidal categories does not have e invertible.) Note that as e is not an isomorphism, we do not have the Eilenberg and Kelly versions of conditions 2 and 4 which are expressed in terms of i; and these conditions fail in our leading examples. Furthermore we have made no explicit naturality assumptions on i and j. It follows from the definition that they can be equipped to be pseudo-natural and again that is all we get in leading examples. The final technical condition has the consequence that the section $Vi_{[A,A]}$ takes the identity 1_A to j_A as required by Eilenberg and Kelly. However, its true significance is that it ensures that a pseudo-closed 2-category is a closed 2-multicategory. We do not elaborate on this point of view here, but it indicates why the details of the equivalence matter, and so why we are careful to work with adjoint retract equivalences. Finally, we observe that we are able to give our definition so that in T-Alg almost all the structure maps are strict maps of T-algebras: in T-Alg, i_A is not strict, but we do not make substantial use of it in the axioms above.

While we develop a general theory in this paper we are mainly interested in pseudo-closed categories which are *symmetric* in the following sense.

Definition 2. A *symmetry* on a pseudo-closed 2-category $\mathscr{K}$ consists of a 2-natural transformation $\tau : [A,[B,C]] \to [B,[A,C]]$ which is an involution (that is, $\tau^2 = \mathrm{id}$) and satisfying

Identity laws

$$\begin{tikzcd}
{[A,[I,C]]} \arrow[rr, "\tau"] \arrow[dr, "{[A,e_C]}"'] & & {[I,[A,C]]} \arrow[dl, "e_{[A,C]}"] \\
& {[A,C]} &
\end{tikzcd}$$

$$\begin{tikzcd}
{[A,[I,C]]} \arrow[rr, "\tau"] & & {[I,[A,C]]} \\
& {[A,C]} \arrow[ul, "i_{[A,C]}"] \arrow[ur, "{[A,i_C]}"'] &
\end{tikzcd}$$

Yang–Baxter law

$$\begin{tikzcd}
& {[A,[C,[B,D]]]} \arrow[r] & {[C,[A,[B,D]]]} \arrow[dr] & \\
{[A,[B,[C,D]]]} \arrow[ur] \arrow[dr] & & & {[C,[B,[A,D]]]} \\
& {[B,[A,[C,D]]]} \arrow[r] & {[B,[C,[A,D]]]} \arrow[ur] &
\end{tikzcd}$$

As we have mentioned, this is not the most general possible notion of (symmetric) pseudo-closedness. At a relatively trivial level, even Eilenberg and Kelly could have asked for an isomorphism between $V[-,-]$ and $\mathscr{K}(-,-)$: by their choice, a monoidal category subject to the usual adjointness condition need not be closed. Here, it might be more natural to ask for a retract equivalence from $V[-,-]$ to $K(-,-)$, with the section only pseudo-natural; but our examples have equality, and the spirit of our approach is to be as strict as the examples allow us. Also we asked for equalities in our conditions 1–5 where one might expect invertible modifications and coherence conditions. But again our examples allow us to be strict and we have not analysed the appropriate coherence conditions.

2.2. An embedding theorem

It appears to be folklore that any closed category embeds in a monoidal closed category preserving the closed structure. This fact shows that the axioms for a closed category are exactly what is true of the closed structure in a monoidal closed category. Presumably, Eilenberg and Kelly were aware of this; but they do not refer to it explicitly. The result in the symmetric case is effectively in Day [6] using his convolution tensor product in presheaf categories [4,5]. Day certainly had the technology for the non-symmetric case which is in Laplaza [14]. Our approach is similar, but there are a number of additional twists. (We know, from Peter Johnstone's diary, that the embedding theorems were also explicitly proved by Freyd in the early 1970s; but his treatment was never published.) To enable us to state an analogous theorem we first introduce a notion of closed functor which parallels that of [7].

Definition 3. A *closed functor* $F : \mathscr{K} \to \mathscr{L}$ of pseudo-closed 2-categories consists of a 2-functor $F : \mathscr{K} \to \mathscr{L}$ equipped with

- a 1-cell $\phi : I \to FI$, and
- a natural transformation $\phi : F[A,B] \to [FA,FB]$,

satisfying the following conditions.

1.
$$\begin{array}{ccc} I & \xrightarrow{j} & [FA,FA] \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ FI & \xrightarrow[Fj]{} & F[A,A] \end{array}$$

2.
$$\begin{array}{ccc} F[I,A] & \xrightarrow{\phi} & [FI,FA] \\ {\scriptstyle Fe}\downarrow & & \downarrow{\scriptstyle [\phi,FA]} \\ FA & \xleftarrow[e]{} & [I,FA] \end{array}$$

3.
$$\begin{array}{ccccc} F[B,C] & \xrightarrow{Fk_A} & F[[A,B],[A,C]] & \xrightarrow{\phi} & [F[A,B],F[A,C]] \\ {\scriptstyle\phi}\downarrow & & & & \downarrow{\scriptstyle [F[A,B],\phi]} \\ [FB,FC] & \xrightarrow[k_{FA}]{} & [[FA,FB],[FA,FC]] & \xrightarrow[[\phi,[FA,FC]]]{} & [F[A,B],[FA,FC]] \end{array}$$

If $\mathscr{K}$ and $\mathscr{L}$ are symmetric, $F : \mathscr{K} \to \mathscr{L}$ is a *closed functor* of symmetric pseudo-closed 2-categories if in addition we have the following.

$$\begin{array}{ccccc} F[A,[B,C]] & \xrightarrow{\phi} & [FA,F[B,C]] & \xrightarrow{[FA,\phi]} & [FA,[FB,FC]] \\ {\scriptstyle F\tau}\downarrow & & & & \downarrow{\scriptstyle\tau} \\ F[B,[A,C]] & \xrightarrow[\phi]{} & [FB,F[A,C]] & \xrightarrow[[FB,\phi]]{} & [FB,[FA,FC]]. \end{array}$$

Theorem 1. *Any small pseudo-closed* 2*-category* $\mathscr{K}$ *embeds via a closed functor* $\Phi : \mathscr{K} \to \mathscr{L}$ *in a monoidal closed* 2*-category* $\mathscr{L}$ *where*

- Φ *is full and faithful*;
- Φ *preserves the internal hom* $[-,-]$ *up to coherent isomorphism in the sense that* $\phi : E[A,B] \to [EA,EB]$ *is an isomorphism.*

If $\mathscr{K}$ *is a symmetric pseudo-closed* 2*-category then* $\mathscr{L}$ *can be taken to be symmetric monoidal closed* 2*-category and* Φ *is a closed functor between symmetric pseudo-closed* 2*-categories.*

Proof. We give just an outline in the non-symmetric case. Let $\mathscr{E}$ be the 2-category of $\mathscr{K}$-endofunctors of $\mathscr{K}$, $\mathscr{K}$-natural transformations and $\mathscr{K}$-modifications: $\mathscr{E}$ is a monoidal 2-category under composition. A Yoneda argument shows that $\mathscr{K}$ embeds in $\mathscr{E}^{\mathrm{op}}$ preserving the function spaces. Let $\mathscr{L}$ be the 2-category $[\mathscr{E}, Cat]$ which is monoidal closed essentially by Day [4]. The Yoneda embedding preserves the tensor product and such function spaces as exist. We let Φ be the composite of the two embeddings, and we are done. $\square$

The result we have just given depends only on the closed 2-multicategory structure. We cannot really detect the section i which is only pseudo-functorial. Our attempts to do so have not given an interesting result. Specifically, we cannot yet make good the line of argument at the end of the next section.

2.3. Pseudo-monoidal structure

Eilenberg and Kelly support their notion of closed category with a result (their Theorem 5.3) to the effect that a closed category $\mathscr{V}$ with suitable left adjoints to $[A,-]: \mathscr{V} \to \mathscr{V}$ is monoidal. We give a corresponding result for our notion of pseudo-closedness. To present it, we need the notion of pseudo-monoidal 2-category. In the literature [8], this is called a monoidal 2-category, but we generally call pseudo-notions pseudo.

Definition 4. A *pseudo-monoidal* 2-category is a 2-category K together with an object I, a pseudo-functor $\otimes : K \times K \to K$, pseudo-natural equivalences with components $a : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l : I \otimes X \to X$, and $r : X \to X \otimes I$, and invertible modifications with components

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "a \otimes W"] \arrow[d, "a"'] & (X \otimes (Y \otimes Z)) \otimes W \arrow[r, "a"] \arrow[d, phantom, "\Downarrow\pi"] & X \otimes ((Y \otimes Z) \otimes W) \arrow[d, "X \otimes a"] \\
(X \otimes Y) \otimes (Z \otimes W) \arrow[rr, "a"'] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

$$\begin{tikzcd}
(I \otimes X) \otimes Y \arrow[r, "a"] \arrow[d, "l \otimes Y"'] \arrow[dr, phantom, "\lambda\Downarrow"] & I \otimes (X \otimes Y) \arrow[dl, "l"] \\
X \otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "r"] \arrow[dr, "X \otimes r"'] & (X \otimes Y) \otimes I \arrow[d, "a"] \arrow[dl, phantom, "\Downarrow\rho"] \\
& X \otimes (Y \otimes I)
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[r, "a"] \arrow[d, "r \otimes Y"'] \arrow[dr, phantom, "\Downarrow\mu"] & X \otimes (I \otimes Y) \arrow[d, "X \otimes l"] \\
X \otimes Y \arrow[r, "\mathrm{id}"'] & X \otimes Y
\end{tikzcd}$$

satisfying three tricategory axioms concerning the equality of pastings. These are as in [8]: so a pseudo-monoidal 2-category is a one object tricategory which is locally a 2-category. To save space we omit the axioms here.

We now present an analogue of Theorem 5.3 of Eilenberg and Kelly [7].

Theorem 2. *Suppose that K is a pseudo-closed 2-category such that for 0-cells A, B we have the 2-functor $[A,[B,-]]: K \to K$ birepresentable in the following internal sense. We have an object $A \otimes B$ and an equivalence*

$$d_C : [A \otimes B, C] \to [A,[B,C]]$$

2-natural in C such that the diagram

$$\begin{tikzcd} {[C,D]} \arrow[r, "k_B"] \arrow[d, "k_{A\otimes B}"'] & {[[B,C],[B,D]]} \arrow[r, "k_A"] & {[[A,[B,C]],[A,[B,D]]]} \arrow[d, "{[d_C,[A,[B,D]]]}"] \\ {[[A\otimes B,C],[A\otimes B,D]]} \arrow[rr, "{[[A\otimes B,C],d_D]}"'] & & {[[A\otimes B,C],[A,[B,D]]]} \end{tikzcd}$$

commutes. Then the 2-category K acquires the structure of a pseudo-monoidal 2-category pseudo-closed in the sense that $- \otimes B$ has a right biadjoint.

Proof. We give the barest outline of the construction of the data. To save space we represent tensor by juxtaposition. Note that by assumption we have a choice of unit $A \to [B, AB]$ strictly dinatural in B. We use it systematically to give most of the data: that is, we give first an exponential form of the data, which itself makes use of a chosen pseudo-natural inverse to d.

Definition of a: Consider the 2-natural

$$A \xrightarrow{\text{unit}} [BC, A(BC)] \xrightarrow{k_C} [[C,BC],[C,A(BC)]] \xrightarrow{[\text{unit},[C,\text{id}]]} [B,[C,A(BC)]].$$

Take $a : (AB)C \to A(BC)$ to be the pseudo-natural transformation corresponding to that exponential form. Representability shows it is a pseudo-natural equivalence.

Definition of l: Take $l : IA \to A$ to correspond to the exponential form $j : I \to [A,A]$. Again by representability this is a pseudo-natural equivalence.

Definition of r: Take $r : A \to AI$ to be the composite

$$A \xrightarrow{\text{unit}} [I, AI] \xrightarrow{e} AI.$$

The equivalence inverse is given by the transpose of $i : A \to [I,A]$.

Definition of λ: We have an isomorphism between

$$I \xrightarrow{j} [AB,AB] \quad \text{and} \quad I \xrightarrow{\text{unit}} [AB, I(AB)] \xrightarrow{[1,l]} [AB,AB].$$

So simple naturality considerations show that

$$I \xrightarrow{j} [AB,AB] \xrightarrow{k} [[B,AB],[B,AB]] \xrightarrow{[\text{unit},[B,AB]]} [A,[B,AB]]$$

is isomorphic to the transpose of $l \cdot a : (IA)B \to AB$. But by pseudo-naturality of j we get an isomorphism with

$$I \xrightarrow{j} [A,A] \xrightarrow{[a,\mathrm{unit}]} [A,[B,AB]],$$

which is the transpose of $lB : (IA)B \to AB$. Composing the isomorphisms and taking a transpose gives λ.

Definition of μ: Consider the diagram

$$\begin{tikzcd}
{[IB, A(IB)]} \arrow[r, "{[1, Al]}"] & {[IB, AB]} \arrow[r, "k"] & {[[B, IB], [B, AB]]} \arrow[r, "{[\mathrm{unit}, 1]}"] & {[I, [B, AB]]} \arrow[r, "e"] & {[B, AB]} \\
A \arrow[u, "\mathrm{unit}"] \arrow[r, "\mathrm{unit}"'] & {[B, AB]} \arrow[u, "{[e, 1]}"] \arrow[r, "k"'] & {[[B, B], [B, AB]]} \arrow[u, "{[[B, e], 1]}"] \arrow[ur, "{[j, 1]}"'] \arrow[u, phantom, shift left=8ex, "\Downarrow"] & &
\end{tikzcd}$$

which commutes up to an invertible 2-cell indicated. The top is the transpose of

$$AB \xrightarrow{rB} (AI)B \xrightarrow{a} A(IB) \xrightarrow{Al} AB,$$

while the bottom is that of the identity. Passing to the transpose gives μ.

Definition of ρ: Consider the commuting diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{unit}"] & {[BI, A(BI)]} \arrow[r, "k"] \arrow[dr, "{[e, 1]}"'] & {[[I, BI], [I, A(BI)]]} \arrow[r, "{[\mathrm{unit}, 1]}"] \arrow[d, "{[e, 1]}"] & {[B, [I, A(BI)]]} \arrow[d, "{[e, 1]}"] \\
& & {[[I, BI], A(BI)]} \arrow[r, "{[\mathrm{unit}, 1]}"'] & {[B, A(BI)]}
\end{tikzcd}$$

One readily finds an isomorphism between the top composite and the transpose of $a \cdot r : AB \to A(BI)$ and an isomorphism between the bottom and the transpose of $Ar : AB \to A(BI)$. Compose and pass to the transpose to get ρ.

Definition of π: To construct π, one first gives a diagram between the relevant transposes which are 1-cells $A \to [B,[C,[D,A(B(CD))]]]$: one fills in largely with commuting diagrams, though one does need an invertible 2-cell. Then one passes to the transpose. We omit the details.

Checking the axioms: In each case, the strategy is to consider the two cells in exponential form, confirm equality of these and deduce the result. We make no attempt to transcribe our rough notes. $\square$

Naturally, we are unhappy with the proof we have just outlined. Since the data we start from is in no way symmetric we expect some messy difficulties: but the calculations we do not give are very tiresome, and it would be only too easy to have made a slip. Hence, we would like a more conceptual proof. One natural line was suggested by the referee: a suitable 'embedding' $\Psi : \mathscr{K} \to \mathscr{L}$ in a monoidal closed 2-category will give a biequivalence between $\mathscr{K}$ and its essential image $\mathscr{M}$ say. Then one should be able to transport pseudo-monoidal structure from $\mathscr{M}$ to $\mathscr{K}$ along the biequivalence (Lemma 3.6 of [8]). This is appealing as it requires less strict conditions than those we have given and it would deal readily with the symmetric case. (According to current thinking, a symmetric pseudo-monoidal bicategory would correspond to a weak 6-category with just one three cell; but whatever the details the structure must transport along biequivalences.) Unfortunately, we do not currently have $\Psi : \mathscr{K} \to \mathscr{L}$ with $I \to \Psi(I)$ an equivalence, so we are not yet able to bring this off.

3. Pseudo-commutativity for a 2-monad

Throughout this section we work with a 2-category $\mathscr{K}$ with finite products. What we do works as well in a monoidal 2-category; but with finite products we feel justified in suppressing associativities. We start with the standard 2-categorical notion of strength: the development mirrors that in Kock [11]. Then we present our axioms for a pseudo-commutativity, derive some basic consequences, and sketch the connection with Kelly's formulation in [10]. Finally, we reformulate the axioms in a way which is convenient for our main construction.

3.1. Strength and enrichment

Suppose that $T : \mathscr{K} \to \mathscr{K}$ is a 2-endofunctor. A *strength* t for T consists of a 2-natural transformation with components

$$t_{A,B} : A \times TB \to T(A \times B)$$

subject to obvious identity and associativity laws.

A strength t corresponds, by symmetry of finite products in $\mathscr{K}$, to a costrength t^*, whose components we denote by

$$t^*_{A,B} : TA \times B \to T(A \times B).$$

In case $\mathscr{K}$ is a cartesian closed 2-category, a strength corresponds to an enrichment

$$T : [A,B] \to [TA,TB]$$

of T in K, satisfying the usual axioms on the nose. The enrichment T is related to t in that the diagrams

$$\begin{tikzcd} A \arrow[r, "in"] \arrow[d, "in"'] & {[B, A \times B]} \arrow[d, "T"] \\ {[TB, A \times TB]} \arrow[r, "{[TB,t]}"'] & {[TB, T(A \times B)]} \end{tikzcd} \qquad \begin{tikzcd} {[A,B] \times TA} \arrow[r, "t"] \arrow[d, "T \times TA"'] & {T([A,B] \times A)} \arrow[d, "Tev"] \\ {[TA,TB] \times TA} \arrow[r, "ev"'] & TB \end{tikzcd}$$

commute. Equally, in this setting, the data of a strength also corresponds to a lifting

$$\bar{t} : T[A,B] \to [A,TB],$$

satisfying natural axioms. $\bar{t}$ is related to t^* in that the diagrams

$$\begin{tikzcd} TA \arrow[r, "Tin"] \arrow[d, "in"'] & T([B,A\times B]) \arrow[d, "\bar{t}"] \\ {[B,TA\times B]} \arrow[r, "{[B,t^*]}"'] & {[B,T(A\times B)]} \end{tikzcd} \qquad \begin{tikzcd} T[A,B]\times A \arrow[r, "t^*"] \arrow[d, "\bar{t}"'] & T([A,B]\times A) \arrow[d, "Tev"] \\ {[A,TB]\times A} \arrow[r, "ev"'] & TB \end{tikzcd}$$

commute.

Now suppose that (T,η,μ) is a 2-monad. Then as well as requiring a strength (or enrichment) for T it is natural to require that η and μ be $\mathscr{K}$-natural transformations. We say then that (T,η,μ) is a *strong* 2-monad with strength t or enrichment T. Observe that in case $\mathscr{K}$ only has products $\mathscr{K}$-naturality of η and μ can be expressed quite simply in terms of the basic strength t. If (T,η,μ) is a strong 2-monad then T can be regarded as a monoidal 2-functor in two distinct ways. The nullary component is in each case given by

$$I \xrightarrow{\eta_I} TI,$$

while the binary component is

$$TA\times TB \xrightarrow{t^*} T(A\times TB) \xrightarrow{Tt} T^2(A\times B) \xrightarrow{\mu_{A\times B}} T(A\times B)$$

on the one hand, and symmetrically

$$TA\times TB \xrightarrow{t} T(TA\times B) \xrightarrow{T(t^*)} T^2(A\times B) \xrightarrow{\mu_{A\times B}} T(A\times B)$$

on the other. Either way, not only is T a monoidal 2-functor, but also η is a monoidal 2-natural transformation. However μ is not generally a monoidal 2-natural transformation, and the basic result is the following.

Theorem 3. *Suppose* (T,η,μ) *is a strong* 2*-monad. Then* (*either of*) *the above makes* (T,η,μ) *into a monoidal* 2*-monad if and only if* (T,η,μ) *is commutative in the sense that the two monoidal structures on* T *coincide.*

Thus far we have the enriched version of Kock's theory, so the proof is as in [11,13]. Note that if (T,η,μ) is commutative, so monoidal, then with suitable completeness and cocompleteness the 2-category of T-algebras and strict algebra maps is a symmetric monoidal closed 2-category in the strict sense.

3.2. Pseudo-commutativity

As Kelly [10] already observed commutative 2-comonads on *Cat* are rare while one can easily find 2-monads which are pseudo-commutative in a suitable sense: we now give one such sense.

Definition 5. A *pseudo-commutativity* for a strong 2-monad (T,η,μ) is an invertible modification

$$\begin{tikzcd} TA\times TB \arrow[r, "t^*"] \arrow[d, "t"'] & T(A\times TB) \arrow[r, "T(t)"] \arrow[d, phantom, "\Downarrow\gamma_{A,B}"] & T^2(A\times B) \arrow[d, "\mu_{A\times B}"] \\ T(TA\times B) \arrow[r, "Tt^*"'] & T^2(A\times B) \arrow[r, "\mu_{A\times B}"'] & T(A\times B) \end{tikzcd}$$

such that the following three strength axioms, two η axioms and two μ axioms hold.

1. $\gamma_{A\times B,C}\cdot(t_{A,B}\times TC)=t_{A,B\times C}\cdot(A\times\gamma_{B,C})$
2. $\gamma_{A,B\times C}\cdot(TA\times t_{B,C})=\gamma_{A\times B,C}\cdot(t^*_{A,B}\times TC)$
3. $\gamma_{A,B\times C}\cdot(TA\times t^*_{B,C})=t^*_{A\times B,C}\cdot(\gamma_{A,B}\times C)$
4. $\gamma_{A,B}\cdot(\eta_A\times TB)$ is an identity modification
5. $\gamma_{A,B}\cdot(TA\times\eta_B)$ is an identity modification
6. $\gamma_{A,B}\cdot(\mu_A\times TB)$ is equal to the pasting

$$\begin{tikzcd}
T^2A\times TB \arrow[r, "t^*"] \arrow[d, "t"'] & T(TA\times TB) \arrow[r, "Tt^*"] \arrow[d, "Tt"] & T^2(A\times TB) \arrow[r, "T^2t"] \arrow[d, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A\times B) \arrow[d, "T\mu_{A\times B}"] \\
T(T^2A\times B) \arrow[d, "Tt^*"'] \arrow[dr, phantom, "\Downarrow \gamma_{TA,B}"] & T^2(TA\times B) \arrow[r, "T^2t^*"'] \arrow[d, "\mu_{TA\times B}"] & T^3(A\times B) \arrow[r, "T\mu_{A\times B}"'] \arrow[d, "\mu_{T(A\times B)}"] & T^2(A\times B) \arrow[d, "\mu_{A\times B}"] \\
T^2(TA\times B) \arrow[r, "\mu_{TA\times B}"'] & T(TA\times B) \arrow[r, "Tt^*"'] & T^2(A\times B) \arrow[r, "\mu_{A\times B}"'] & T(A\times B)
\end{tikzcd}$$

7. $\gamma_{A,B}\cdot(TA\times\mu_B)$ is equal to the pasting

$$\begin{tikzcd}
TA\times T^2B \arrow[r, "t^*"] \arrow[d, "t"'] & T(A\times T^2B) \arrow[r, "Tt"] \arrow[d, phantom, "\Downarrow \gamma_{A,TB}"] & T^2(A\times TB) \arrow[d, "\mu_{(A\times TB)}"] \\
T(TA\times TB) \arrow[r, "Tt^*"'] \arrow[d, "Tt"'] & T^2(A\times TB) \arrow[r, "\mu_{A\times TB}"'] \arrow[d, "T^2t"] & T(A\times TB) \arrow[d, "Tt"'] \\
T^2(TA\times B) \arrow[d, "T^2t^*"'] \arrow[dr, phantom, "\Downarrow T\gamma_{A,B}"] & T^3(A\times B) \arrow[r, "\mu_{T(A\times B)}"] \arrow[d, "T\mu_{A\times B}"] & T^2(A\times B) \arrow[d, "\mu_{A\times B}"] \\
T^3(A\times B) \arrow[r, "T\mu_{A\times B}"'] & T^2(A\times B) \arrow[r, "\mu_{A\times B}"'] & T(A\times B)
\end{tikzcd}$$

There may appear to be some redundancy in the definition, and if our commutativity is symmetric in the sense of Section 3.6, that is indeed the case. However, without the symmetry condition, the only apparent redundancy is given by the following result.

Proposition 1. *Any two of the strength axioms implies the third.*

If the modification γ in our definition were an identity, T would be a commutative 2-monad and the axioms would be redundant. But in our leading example, γ is not an identity but rather is determined by a non-trivial symmetry. To aid understanding we next present an outline of it.

3.3. The monad for symmetric strict monoidal categories

Our leading example is that of symmetric strict monoidal categories: it has specific applications in theoretical computer science. While the monad in question is not a completely typical pseudo-commutative monad, it is simple, and one can derive a feel for the behaviour of a general pseudo-commutativity from it. For this section, let T be the 2-monad for symmetric strict monoidal categories.

- Given a category A, the category TA has as objects sequences $a_1 \ldots a_n$ of objects of A (with maps generated by symmetries and the maps of A); the tensor product is concatenation.
- Given two categories A and B, the category $TA \times TB$ has as objects pairs $((a_1 \ldots a_n), (b_1, \ldots b_m))$; and the two maps $TA \times TB \to T(A \times B)$ take such pairs to the sequences of all (a_i, b_j) ordered according to the two possible lexicographic orderings. In fact,
$$TA \times TB \xrightarrow{t^*} T(TA \times B) \xrightarrow{T(t)} T^2(A \times B) \xrightarrow{\mu_{A\times B}} T(A \times B)$$
gives the ordering $(a_1, b_1), (a_1, b_2), \ldots$ in which the first coordinate takes precedence, while
$$TA \times TB \xrightarrow{t} T(TA \times B) \xrightarrow{T(t^*)} T^2(A \times B) \xrightarrow{\mu_{A\times B}} T(A \times B)$$
gives the ordering $(a_1,\ b_1), (a_2, b_1), \ldots$ in which the second coordinate takes precedence.
- The component $\gamma_{A,B}$ of the modification is given by the unique symmetry mediating between the two lexicographic orders.

Thus, one should regard the pseudo-commutativity γ as a higher level symmetry. It has properties analogous to those of a symmetry in a monoidal category; and though we shall not here spell these out in detail, we shall signal the fact that they are at issue by referring to γ as a mediating symmetry.

We now indicate the force of our various axioms.

- The strength axioms concern the various lexicographic orderings of the sequences (a_i, b_j, c_k) where there is just one a_i (or b_j or c_k). Various orderings are identified and as a result there are prima facie two processes for mediating between the orderings: these are equal. So the axioms reflect the fact that there is a unique way to mediate between a pair of orderings.
- The η axioms are easy to understand: they express the fact that the two lexicographic orderings of the (a_i, b_j) are equal if one of n or m is 1.
- The μ axioms take more explaining. Take a sequence $a^1, \ldots, a^n$ of sequences $a^i_1, \ldots, a^i_{m(i)}$. Concatenation gives a sequence a^i_j where the order is determined by the precedence (i, j): that is, i takes precedence over j. Take this concatenated sequence together with a sequence $b_1, \ldots, b_p$: then the 2-cell $\gamma_{A,B} \cdot (\mu_A \times TB)$ mediates between the order on the (a^i_j, b_k) with precedence (i, j, k) and that with precedence (k, i, j). However we can also use $\mu \cdot T\gamma_{A,B} \cdot t^*$ to mediate between the orders determined by (i, j, k) and (i, k, j); and $\mu \cdot Tt^* \cdot \gamma_{TA,B}$ to mediate between the orders determined by (i, k, j) and (k, i, j). Composing these two gives the first. So again the issue is the unique way to mediate between a pair of orderings.

These considerations point to coherence phenomena which we briefly discuss but do not treat formally in Section 3.5.

3.4. Elementary consequences of the axioms

We need to understand some consequences of our axioms. We consider 2-cells between 1 cells with domain $TA \times TB \times TC$ and codomain $T(A \times B \times C)$. In writing these and further identities we generally leave off subscripts which can be read off from other subscripts by type checking.

Proposition 2. *The following identities hold between* 2*-cells of the above form.*

(i) $\mu \cdot T(\gamma_{A\times B,C}) \cdot t^* \cdot Tt \times TC \cdot t^* \times TC = \mu \cdot Tt \cdot T(A \times \gamma_{B,C}) \cdot t^*,$

$\mu \cdot T(\gamma_{A,B\times C}) \cdot t \cdot TA \times Tt^* \cdot TA \times t = \mu \cdot Tt^* \cdot T(\gamma_{A,B} \times C) \cdot t,$

(ii) $\mu \cdot T(\gamma_{A\times B,C}) \cdot t^* \cdot Tt^* \times TC \cdot t \times TC = \mu \cdot T(\gamma_{A,B\times C}) \cdot t \cdot TA \times Tt \cdot TA \times t^*,$

(iii) $\mu \cdot Tt^* \cdot \gamma_{T(A\times B),C} \cdot Tt \times TC \cdot t^* \times TC = \mu \cdot Tt \cdot \gamma_{A,T(B\times C)} \cdot TA \times Tt^* \cdot TA \times t,$

(iv) $\mu \cdot Tt^* \cdot \gamma_{T(A\times B),C} \cdot Tt^* \times TC \cdot t \times TC = \mu \cdot Tt^* \cdot t \cdot TA \times \gamma_{B,C},$

$\mu \cdot Tt \cdot \gamma_{A,T(B\times C)} \cdot TA \times Tt \cdot TA \times t^* = \mu \cdot Tt \cdot t^* \cdot \gamma_{A,B} \times TC.$

These identities exhibit different ways to construct some of the simplest mediating symmetries. We have saved space by omitting the diagrams which give the proofs, but note that alternative expression for these symmetries can be read off from them. For example (iii) is also equal to

$$\begin{aligned}&\mu \cdot Tt^* \cdot T(t \times C) \cdot \gamma_{A\times TB,C} \cdot t^* \times TC\\ &\quad = \mu \cdot Tt \cdot T(A \times t^*) \cdot \gamma_{A,TB\times C} \cdot TA \times t.\end{aligned}$$

Using the μ axioms we immediately deduce some simple pasting identities.

Proposition 3. *The following identities hold, for* 2*-cells of the above form.*

$$\begin{aligned}&\gamma_{A,B\times C} \cdot TA \times \mu \cdot TA \times Tt^* \cdot TA \times t\\ &\quad = (\mu \cdot Tt^* \cdot t \cdot \gamma_{A,B} \times TC)(\mu \cdot Tt^* \cdot T(t \times C) \cdot \gamma_{A\times TB,C} \cdot t^* \times TC)\\ &\quad = (\mu \cdot Tt^* \cdot t \cdot \gamma_{A,B} \times TC)(\mu \cdot Tt \cdot T(A \times t^*) \cdot \gamma_{A,TB\times C} \cdot TA \times t),\end{aligned}$$

$$\begin{aligned}&\gamma_{A,B\times C} \cdot TA \times \mu \cdot TA \times Tt \cdot TA \times t^*\\ &\quad = (\mu \cdot T\gamma_{A\times B,C} \cdot t^* \cdot Tt^* \times TC \cdot t \times TC)(\mu \cdot Tt \cdot t^* \cdot \gamma_{A,B} \times TC),\end{aligned}$$

$$\begin{aligned}&\gamma_{A\times B,C} \cdot \mu \times TC \cdot Tt^* \times TC \cdot t \times TC\\ &\quad = (\mu \cdot Tt^* \cdot t \cdot TA \times \gamma_{B,C})(\mu \cdot T\gamma_{A,B\times C} \cdot t \cdot TA \times Tt \cdot TA \times t^*)\\ &\quad = (\mu \cdot Tt^* \cdot t \cdot TA \times \gamma_{B,C})(\mu \cdot T\gamma_{A\times B,C} \cdot t^* \cdot Tt^* \times TC \cdot t \times TC)\end{aligned}$$

$$\gamma_{A\times B,C}\cdot\mu\times TC\cdot Tt\times TC\cdot t^*\times TC$$
$$=(\mu\cdot Tt\cdot TA\times Tt^*\cdot\gamma_{A,TB\times C}\cdot TA\times t)(\mu\cdot Tt\cdot t^*\cdot TA\times\gamma_{B,C})$$
$$=(\mu\cdot Tt^*\cdot T(t\times C)\cdot\gamma_{A\times TB,C}\cdot t^*\times TC)(\mu\cdot Tt\cdot t^*\cdot TA\times\gamma_{B,C}).$$

One can regard this as essentially providing the Mac Lane coherence hexagon conditions for our mediating symmetry, but we do not make that precise here.

To indicate the force of our axioms we prove at once the following associativity equation.

Proposition 4. $\gamma_{A,B\times C}\cdot(TA\times\gamma_{B,C})=\gamma_{A\times B,C}\cdot(\gamma_{A,B}\times TC)$.

Proof. We can choose one of a number of manipulations. For example, we can write the left-hand side as the pasting

$$(\gamma_{A,B\times C}\cdot(TA\times\mu)\cdot(TA\times t^*)\cdot(TA\times t))(\mu\cdot Tt\cdot t^*\cdot(TA\times\gamma_{B,C}))$$
$$=(\mu\cdot Tt\cdot TA\times T\gamma_{B,C}\cdot t^*)(\mu\cdot Tt\cdot T(A\times t^*)\cdot\gamma_{A,TB\times C}\cdot TA\times t)$$
$$(\mu\cdot Tt\cdot t^*\cdot(TA\times\gamma_{B,C})).$$

Similarly the right-hand side is the pasting

$$(\mu\cdot Tt^*\cdot t\cdot\gamma_{A,B}\times TC)(\gamma_{A\times B,C}\cdot\mu\times TC\cdot Tt\times TC\cdot t^*\times TC)$$
$$=(\mu\cdot Tt^*\cdot t\cdot\gamma_{A,B}\times TC)(\mu\cdot Tt^*\cdot T(t\times C)\cdot\gamma_{A\times TB,C}\cdot t^*\times TC)$$
$$=(\mu\cdot Tt\cdot t^*\cdot(TA\times\gamma_{B,C})).$$

These are equal by an equation above. □

The 'associativity equation' is not great terminology: for a mediating symmetry it corresponds to the standard Artin braid identities, and the proof above parallels the derivation of the braid identities from the coherence hexagon. The equation is an identity of 2-cells between 1-cells $TA\times TB\times TC\to T(A\times B\times C)$. One can readily check that precomposing it with η_A,η_B and η_C give the three strength conditions.

Proposition 5. *The three strength axioms for a pseudo-commutativity are equivalent in the presence of the η and μ axioms to the single associativity equation.*

3.5. Coherence

The associativity equation is a manifestation of a coherence phenomenon which we briefly describe. From a strong monad we can construct maps $TA_1\times\cdots\times TA_n\to T(A_1\times\cdots\times A_n)$ in many different ways. First, the strength gives a unique map

$$t_i: A_1\times\cdots\times A_{i-1}\times TA_i\times A_{i+1}\times\cdots\times A_n\to T(A_1\times\cdots\times A_n).$$

Then for ρ a permutation on $\{1,\ldots,n\}$, we get a 1-cell

$$t_\rho: TA_1\times\cdots\times TA_n\to T(A_1\times\cdots\times A_n)$$

where

$$t_\rho = (\cdots \mu \cdot \mu \cdot \mu) \cdot (\cdots T^2 t_{\rho(3)} \cdot Tt_{\rho(2)} \cdot t_{\rho(1)}).$$

Generally the t_ρ are distinct for distinct ρ. Moreover it is easy to show that any 1-cell $TA_1 \times \cdots \times TA_n \to T(A_1 \times \cdots \times A_n)$ definable from the data for a strong monad can be rewritten in the above form. Thus, we can construct exactly $n!$ distinct 1-cells t_ρ; these correspond to the $n!$ different variants on the lexicographic ordering. We can associate with each of these a unique vertex in a directed Cayley graph for a standard presentation of the symmetric group: we direct edges so that the identity vertex is a source and the reverse list vertex a sink. Then there is a coherence result which concerns 2-cells which can be constructed by composing *positive* versions of γ. (That is, we do not use its inverse, and also do not use the symmetry in $\mathscr{K}$.)

Theorem 4. *We can compose positive versions of γ to give a two cell between t_ρ and t_σ just when there is a directed path in the Cayley graph from ρ to σ. Any two such composites give equal* 2*-cells.*

We leave for a later occasion a precise statement and proof of this coherence result: we do not need it in this paper. However we shall use and so prove a special case of it. There are two generally distinct maps

$$\mu \cdot Tt_j \cdot t_i\colon A_1 \times \cdots \times TA_i \times \cdots \times TA_j \times \cdots \times A_n \to T(A_1 \times \cdot \times A_n)$$

and

$$\mu \cdot Tt_i \cdot t_j\colon A_1 \times \cdots \times TA_i \times \cdots \times TA_j \times \cdots \times A_n \to T(A_1 \times \cdot \times A_n)$$

with our customary codomain $T(A_1 \times \cdots \times A_n)$. If as the notation suggests $i < j$ then we can use γ to give a 2-cell

$$\mu \cdot Tt_j \cdot t_i \to \mu \cdot Tt_i \cdot t_j.$$

A typical construction can be informally described as follows. We use the strength and costrength to drive the occurrences of T together, so that after a 1-cell we have something of form

$$\cdots T(\cdots A_i \cdots) \times T(\cdots A_j \cdots) \cdots \ .$$

We then insert a 2-cell of form $\cdots \gamma \cdots$ with the above as domain of the 1-cells and

$$\cdots T(\cdots A_i \cdots A_j \cdots) \cdots$$

as codomain. Then we use the strength and costrength again to move the T to the outside. So we have a whiskering of an occurrence of γ. The different possibilities depend on where the two copies of T are brought together and how much from the left of A_i and right of t_j has been incorporated at the time γ occurs. Now the strength axioms are exactly what one needs to give equality between neighbouring points in this grid of possibilities.

Theorem 5. *There is a unique* 2-*cell*

$$\gamma_{i,j} : \mu \cdot Tt_j \cdot t_i \to \mu \cdot Tt_i \cdot t_j$$

constructed using only positive versions of γ.

3.6. Symmetric pseudo-commutativities

There is a further property of a pseudo-commutativity which is an obvious feature in our leading example: the two possible lexicographic orderings of the (a_i, b_j) are interchanged by the symmetry on *Cat* in a unique way.

Definition 6. A pseudo-commutativity γ is *symmetric* just when it satisfies the following property:

- $Tc_{A,B} \cdot \gamma_{A,B} \cdot c_{TB,TA}$ is the inverse of $\gamma_{B,A}$.

The uniqueness of the interchange of lexicographic orderings leads one to expect a further coherence phenomenon, now of a more traditional form using both γ and its inverse.

Theorem 6. *Suppose that* γ *is a symmetric pseudo-commutativity. Then we can compose versions of* γ *and* γ^{-1} *to give two cells between any* t_ρ. *Any two such composites give equal* 2-*cells.*

When describing coherence for a general pseudo-commutativity, we excluded use of the symmetry τ, but with symmetry we no longer need do so. Again, we do not give precise details here: however we give some sense of what is involved with a result which is another analogue of a standard axiom for associativity.

Proposition 6. *Let* γ *be a symmetric pseudo-commutativity. Then the* 2-*cell* $\gamma_{A\times B,C} \cdot (\mu_{A\times B} \cdot Tt^* \cdot t) \times TC$ *equals the pasting of*

$$\mu_{A\times B\times C} \cdot Tt^* \cdot t \cdot (TA \times \gamma_{B,C})$$

with

$$T(A \times c) \cdot \mu_{A\times B\times C} \cdot Tt^* \cdot T(\gamma_{A,C} \times B) \cdot t \cdot TA \times Tc \cdot TA \times t^*.$$

Proof. By Proposition 3, it suffices to show that the second 2-cell is equal to

$$\mu \cdot T\gamma_{A,B\times C} \cdot t \cdot TA \times Tt \cdot TA \times t^*.$$

Naturalities and the symmetry axiom reduces this to Axiom 3 of Definition 5. □

We take it that this is the hexagonal axiom relating $\gamma_{A\times B,C}$ to $\gamma_{A,C}$ and $\gamma_{B,C}$ to which Kelly [10] refers. This result gives a taste of the connection between our approach and Kelly's. Using similar arguments we can establish a result of the following form.

Theorem 7. *To give a symmetric pseudo-commutativity on T as above is to give T the structure of a symmetric pseudo-monoidal monad.*

We leave the details of the definition of *symmetric pseudo-monoidal monad* for another occasion, but note that it relates to the form of the definition sketched by Kelly: he more or less defines γ to be a pseudo-commutativity just when it gives rise to a pseudo-monoidal monad. Thus Theorem 7 shows that our axioms are equivalent to the ones Kelly [10] intends. (We have reconstructed Kelly's axioms, and checked this directly.) Furthermore, Kelly's main result is a consequence of Theorem 7.

Theorem 8. *Let T be a 2-monad on Cat equipped with a symmetric pseudo-commutativity γ. Then T lifts to a 2-monad on the 2-category SymMon.*

In [10] Kelly said that he believed that his axioms reduced to a small number. This is certainly the case in our formulation. If γ is symmetric then the three conditions regarding strength are equivalent, the two η conditions are equivalent and the two μ conditions are equivalent.

Proposition 7. *To show that an invertible modification γ as above is a symmetric pseudo-commutativity it suffices to check the symmetry axiom together with one strength axiom, one η axiom, and one μ axiom.*

This renders entirely manageable the problem of showing that we have symmetric pseudo-commutativities in the cases of interest to us. We give a list of examples based on Kelly [10], but omitting cases where one has a strict commutativity.

1. Symmetric strict monoidal categories.
2. Symmetric monoidal categories.
3. Categories with strictly associative finite products (or dually coproducts).
4. Categories with finite products (or dually coproducts).
5. Categories with strictly associative finite biproducts.
6. Categories with finite biproducts.
7. Categories with an action of a symmetric strictly associative monoidal category.
8. Symmetric strict monoidal categories with a strict monoidal endofunctor.
9. Symmetric monoidal categories with a strong monoidal endofunctor.

(Examples intermediate between 3 and 5 occur naturally when considering forms of wiring diagrams [9].) In all the cases listed one can establish the pseudo-commutativity directly without reference to the corresponding clubs.

A general pseudo-commutativity is a mediating symmetry satisfying braid conditions. We believe that there is a non-symmetric pseudo-commutativity on the 2-monad for braided monoidal categories. We do not investigate that here: our current interest (see [9]) is in symmetric examples.

3.7. The exponential transpose

Given a modification γ as above we define a modification $\bar{\gamma}$ using the closed structure as follows. We let $\bar{\gamma}$ be the composite

$$T[A,B] \to [TA, T[A,B] \times TA] \;\overset{\longrightarrow}{\underset{\longrightarrow}{(\Downarrow)}}\; [TA, T([A,B] \times A)] \to [TA, TB]$$

where $(\Downarrow) = [TA, \gamma_{[A,B],A}]$. Conversely we can recover γ from $\bar{\gamma}$ as the composite

$$TA \times TB \to T[B, A \times B] \times TB \;\overset{\longrightarrow}{\underset{\longrightarrow}{(\Downarrow)}}\; [TB, T(A \times B)] \times TB \to T(A \times B)$$

where $(\Downarrow) = \bar{\gamma}_{B,A\times B} \times TB$. So γ and $\bar{\gamma}$ are exponential transposes of each other. The following proposition follows from routine arguments involving the closed structure, using the diagrams given in Section 3.1 which connect t with T and t^* with $\bar{t}$.

Proposition 8. *To give a pseudo-commutativity γ in a cartesian closed 2-category is, by exponentiation, to give an invertible modification*

$$\begin{tikzcd}
T[A,B] \arrow[r, "T(T)"] \arrow[d, "\bar{t}"'] & T[TA,TB] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\bar{\gamma}_{A,B}\Downarrow"] & {[TA,T^2B]} \arrow[d, "{[TA,\mu_B]}"] \\
{[A,TB]} \arrow[r, "T"'] & {[TA,T^2B]} \arrow[r, "{[TA,\mu_B]}"'] & {[TA,TB]}
\end{tikzcd}$$

such that the following conditions hold.

1. $[TA, \bar{\gamma}_{B,C}] \cdot T$ *is the exponential transpose of* $[\gamma_{A,B}, TC] \cdot T$, *where* γ *is obtained from* $\bar{\gamma}$ *as above.*
2. $[A, \bar{\gamma}_{B,C}] \cdot \bar{t}$ *is the exponential transpose of* $[t, TC] \cdot \bar{\gamma}_{A\times B,C}$.
3. $[TA, \bar{t}] \cdot \bar{\gamma}_{A,[B,C]}$ *is the exponential transpose of* $[t^*, TC] \cdot \bar{\gamma}_{A\times B,C}$.
4. $\bar{\gamma}_{A,B} \cdot \eta_{[A,B]}$ *is an identity modification.*
5. $[\eta_A, TB] \cdot \bar{\gamma}_{A,B}$ *is an identity modification.*
6. $\gamma_{A,B} \cdot \mu_{[A,B]}$ *is equal to the pasting*

$$\begin{tikzcd}
T^2[A,B] \arrow[r, "T^2(T)"] \arrow[d, "T(\bar{t})"'] & T^2[TA,TB] \arrow[r, "T(\bar{t})"] \arrow[d, phantom, "T\bar{\gamma}_{A,B}\Downarrow"] & T[TA,T^2B] \arrow[d, "{T[TA,\mu_B]}"] \\
T[A,TB] \arrow[r, "T(T)"'] \arrow[d, "\bar{t}"'] & T[TA,T^2B] \arrow[r, "{T[TA,\mu_B]}"'] \arrow[d, "\bar{t}"] & T[TA,TB] \arrow[d, "\bar{t}"] \\
{[A,T^2B]} \arrow[d, "T"'] \arrow[r, phantom, "\Downarrow\bar{\gamma}_{A,TB}"] & {[TA,T^3B]} \arrow[r, "{[TA,T\mu]}"'] \arrow[d, "{[TA,\mu]}"] & {[TA,T^2B]} \arrow[d, "{[TA,\mu]}"] \\
{[TA,T^3B]} \arrow[r, "{[TA,\mu]}"'] & {[TA,T^2B]} \arrow[r, "{[TA,\mu]}"'] & {[TA,TB]}
\end{tikzcd}$$

7. $[\mu_A, TB] \cdot \bar{\gamma}_{A,B}$ *is equal to the pasting*

$$\begin{tikzcd}
T[A,B] \arrow[r, "T(T)"] \arrow[d, "\bar{t}"'] & T[TA,TB] \arrow[r, "T(T)"] \arrow[d, "\bar{t}"] & T[T^2A,T^2B] \arrow[r, "\bar{t}"] \arrow[d, phantom, "\Downarrow \bar{\gamma}_{TA,TB}"] & {[T^2A,T^3B]} \arrow[d, "{[T^2A,\mu]}"] \\
{[A,TB]} \arrow[d, "T"'] \arrow[r, phantom, "\Downarrow \bar{\gamma}_{A,B}"] & {[TA,T^2B]} \arrow[r, "T"] \arrow[d, "{[TA,\mu]}"] & T[T^2A,T^3B] \arrow[r, "{[T^2A,\mu]}"] \arrow[d, "{[T^2A,T\mu]}"] & {[T^2A,T^2B]} \arrow[d, "{[T^2A,\mu]}"] \\
{[TA,T^2B]} \arrow[r, "{[TA,\mu]}"'] & {[TA,TB]} \arrow[r, "T"'] & {[T^2A,T^2B]} \arrow[r, "{[T^2A,\mu]}"'] & {[T^2A,TB]}
\end{tikzcd}$$

We have numbered the conditions on $\bar{\gamma}$ so that they exactly correspond to the conditions in Definition 5. Hence, though it is not obvious in this form, any two strength axioms imply the third.

One can transpose the associativity equation to get a condition in terms of both $\bar{\gamma}$ and γ.

Proposition 9. *In the presence of the η and μ axioms, the strength axioms are equivalent to the statement that the composite $[TA, \bar{\gamma}_{B,C}] \cdot \bar{\gamma}_{A,[B,C]}$ is the exponential transpose of the composite $[\gamma_{A,B}, TC] \cdot \bar{\gamma}_{A\times B,C}$.*

The formulation in this section relates to a notion of *pseudo-closed monad*. (Note however that it is not simple to express the symmetry condition in terms of the transpose $\bar{\gamma}$.)

4. The 2-category of T-algebras

Our main object of study is the 2-category T-Alg of strict T-algebras and pseudo-maps of algebras for a 2-monad T on Cat. We use the theory of T-Alg as developed in [3]. It is clear that we could work (as in [3]) with a more general base 2-category than Cat, but we do not make that added generality explicit. We write $\mathscr{A} = (A, a)$ for a typical T-algebra: $a : TA \to A$ is the structure map. A *pseudo-map* $(f, \bar{f}) : \mathscr{A} \to \mathscr{B}$ is given by data

$$\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

where the invertible 2-cell $\bar{f}$ satisfies η and μ conditions. (Note that what we call pseudo-maps [3] call morphisms.) We write $f = (f, \bar{f}) : \mathscr{A} \to \mathscr{B}$ for such a pseudo-map, the 2-cell usually being understood.

We aim to give a pseudo-closed structure on the 2-category T-Alg when T is equipped with a pseudo-commutativity. This section contains preliminary material. We recall some of the structure of the 2-category $T\text{-}Alg$ as developed in [3]. We require details of the biadjunction between $T\text{-}Alg$ and Cat, and we need to observe that $T\text{-}Alg$ has products, inserters, and equifiers, and therefore also cotensors, and that these limits lift from Cat. Finally, we develop a little general theory concerning cotensors in the presence of a pseudo-commutativity.

Notation: Generally, we shall use square brackets to denote various kinds of function space. Hence for categories X and Y we write $Cat(X,Y)=[X,Y]$ when we think of the category of functors from X to Y as the cartesian closed structure in Cat; we write $[X,\mathscr{A}]$ for the cotensor of a T-algebra $\mathscr{A}$ over the category X; and we shall write $[\mathscr{A},\mathscr{B}]$ for the pseudo-closed structure on $T\text{-}Alg$ which we introduce much later. This should not cause confusion, but the reader will need to distinguish the cotensor $[A,\mathscr{B}]$ from the function space $[\mathscr{A},\mathscr{B}]$.

4.1. Background from Blackwell–Kelly–Power

We briefly discuss some basic material from [3]. We write $T\text{-}Alg_s$ for the usual Eilenberg–Moore 2-category of algebras and strict maps, and $J : T\text{-}Alg_s \to T\text{-}Alg$ for the locally fully faithful inclusion. We have the standard adjunction

$$F_s \dashv U_s : T\text{-}Alg_s \to Cat.$$

Then we write

$$U : T\text{-}Alg \to Cat$$

for the forgetful functor on $T\text{-}Alg$, and

$$F = J \cdot F_s : Cat \to T\text{-}Alg$$

for the free functor into $T\text{-}Alg$.

In the case when T has rank, flexibility considerations from [3] show that for any category X and T-algebra $\mathscr{A}$ we have a retract equivalence

$$[X,A] = Cat(X,A) \cong T\text{-}Alg_s(FX,\mathscr{A}) \underset{\leftarrow}{\overset{\rightarrow}{\top}} T\text{-}Alg_s((FX)',\mathscr{A}) \cong T\text{-}Alg(FX,\mathscr{A})$$

in Cat. Even without the assumption of rank one can describe the adjoint functors in the equivalence $T\text{-}Alg(FX,\mathscr{A}) \simeq Cat(X,U\mathscr{A})$ directly. We take

$$\bar{e}_{X,\mathscr{A}} = (T\text{-}Alg(FX,\mathscr{A}) \xrightarrow{U} Cat(TX,A) \xrightarrow{Cat(\eta,A)} Cat(X,A));$$

and

$$\bar{i}_{X,\mathscr{A}} = (Cat(X,A) \xrightarrow[F_s \dashv U_s]{\cong} T\text{-}Alg_s(F_sX,\mathscr{A}) \xrightarrow{J} T\text{-}Alg(FX,\mathscr{A})).$$

(The notation may seem rather eccentric, but we are going to lift these maps to T-Alg.) One easily sees that there are adjunctions $\bar{i}_{X,\mathscr{A}} \dashv \bar{e}_{X,\mathscr{A}}$ with

- unit $1_{Cat(X,A)} \to \bar{e}_{X,\mathscr{A}} \cdot \bar{i}_{X,\mathscr{A}}$ equal to the identity, and
- counit $1_{T\text{-}Alg(FX,\mathscr{A})} \to \bar{i}_{X,\mathscr{A}} \cdot \bar{e}_{X,\mathscr{A}}$ invertible,

so that each $\bar{i}_{X,\mathscr{A}} \dashv \bar{e}_{X,\mathscr{A}}$ is an adjoint retract and in particular we have

$$\bar{e}_{X,\mathscr{A}} \cdot \bar{i}_{X,\mathscr{A}} = \mathrm{id} \quad \text{and} \quad \bar{i}_{X,\mathscr{A}} \cdot \bar{e}_{X,\mathscr{A}} \cong \mathrm{id}.$$

It is also easy to check that $\bar{e}$ and $\bar{i}$ are natural in X, that $\bar{e}$ is natural in $\mathscr{A}$ and that $\bar{i}$ is pseudo-natural in $\mathscr{A}$.

The 2-categorical limits of importance to us are the PIE limits characterised by Power and Robinson [16]. It is crucial that (under the usual conditions) T-Alg has all such limits. We give a brief review of these based on the treatment in [3]. Suppose that $\mathscr{K}$ is an arbitrary 2-category. We assume that it is clear what is meant by saying that $\mathscr{K}$ has products, but give the definitions of iso-inserter and equifier.

Definition 7. Given parallel 1-cells $f, g : A \to B$ in $\mathscr{K}$, an (*iso-*)*inserter* is a 1-cell $p : E \to A$ together with an (invertible) 2-cell $\alpha : fp \Rightarrow gp$ such that for any F, composition with k induces an isomorphism between $\mathscr{K}(F,E)$ and the category of cones as above with vertex F.

Concretely, the universality condition means that for any 1-cell $h : F \to A$ and isomorphic 2-cell $\beta : fh \Rightarrow gh$, there exists a unique 1-cell $j : F \to E$ such that $pj = h$ and $\alpha \cdot j = \beta$. We further demand that a corresponding 2-dimensional condition holds.

Definition 8. Given parallel 2-cells $\alpha, \beta : f \Rightarrow g : A \to B$, an *equifier* of α and β is a 1-cell $p : E \to A$ such that $\alpha \cdot p = \beta \cdot p$ and such that for any F, composition with p induces an isomorphism between $\mathscr{K}(F,E)$ and the category of cones as above with vertex F.

The universality condition on k means that for any $h : F \to A$ such that $\alpha \cdot h = \beta \cdot h$, there exists a unique 1-cell $j : F \to E$ such that $pj = h$. Again we further demand that a corresponding two-dimensional condition holds.

From [3] we have the following results on the existence of PIE limits.

Proposition 10. (i) *T-Alg has products and these are preserved by the forgetful 2-functor $U : T\text{-}Alg \to Cat$. Furthermore, the product projections are strict maps, and collectively postcomposition with them reflects strictness.*

(ii) *T-Alg has (iso-)inserters and these are preserved by $U : T\text{-}Alg \to Cat$. The structural 1-cell k is a strict map of algebras and postcomposition with k reflects strictness of algebra maps.*

(iii) *T-Alg has equifiers, and these are preserved by* $U : T\text{-}Alg \to Cat$. *The structural* 1-*cell k is a strict map of algebras and postcomposition with k reflects strictness of algebra maps*.

We also need to understand cotensors.

Definition 9. Suppose that $X \in Cat$ and $A \in \mathscr{K}$. A cotensor of A over X is an object $[X,A] \in \mathscr{K}$ together with a 1-cell $X \to \mathscr{K}([X,A],A)$ in Cat universal amongst such in the sense that for any B in $\mathscr{K}$ it gives

$$X \times \mathscr{K}(B,[X,A]) \to \mathscr{K}([X,A],A) \times \mathscr{K}(B,[X,A]) \to \mathscr{K}(B,A),$$

inducing an isomorphism

$$\mathscr{K}(B,[X,A]) \to Cat(X,\mathscr{K}(B,A)).$$

By [3] it follows from Proposition 10 that the 2-category T-Alg has cotensors, but we need to flesh that out with some details.

Proposition 11. *T-Alg has cotensors, and they are preserved by the forgetful* 2-*functor to Cat. Specifically we have the following*.

(i) *For any T-algebra* $\mathscr{B} = (B,b)$ *and for any small category X, the category* $[X,B]$ *possesses a T-algebra structure, with algebra map given by*

$$T[X,B] \xrightarrow{\bar{t}} [X,TB] \xrightarrow{[X,b]} [X,B].$$

This is the cotensor of $\mathscr{B}$ *by X and we write it as* $[X,\mathscr{B}]$.

(ii) *Composition with any functor* $f : Y \to X$ *induces a strict map*

$$[f,\mathscr{B}] : [X,\mathscr{B}] \to [Y,\mathscr{B}]$$

of T-algebras.

(iii) *Composition with any pseudo-map* $f = (f,\bar{f}) : \mathscr{B} \to \mathscr{C}$ *of T-algebras induces a pseudo-map of T-algebras*

$$[X,f] : [X,\mathscr{B}] \to [X,\mathscr{C}];$$

and this map is strict whenever f is strict.

4.2. The canonical section

Given a small category X and T-algebra $\mathscr{B}$, composition with η_X induces a strict map of T-algebras $[TX,\mathscr{B}] \to [X,\mathscr{B}]$. If T is pseudo-commutative this map has a section.

Proposition 12. *Given a T-algebra $\mathscr{B} = (B, b)$ and a small category X, the composite*

$$[X, B] \xrightarrow{T} [TX, TB] \xrightarrow{[TX,b]} [TX, B]$$

together with the 2-cell

$$\begin{tikzcd}
T[X,B] \arrow[rr, "T(T)"] \arrow[dd, "\bar{t}"'] \arrow[ddr, phantom, "\Downarrow\bar{\gamma}"] & & T[TX,TB] \arrow[r, "T[TX,b]"] \arrow[d, "\bar{t}"] & T[TX,B] \arrow[d, "\bar{t}"] \\
 & & {[TX,T^2B]} \arrow[r, "{[TX,Tb]}"] \arrow[d, "{[TX,\mu_B]}"] & {[TX,TB]} \arrow[dd, "{[TX,b]}"] \\
{[X,TB]} \arrow[r, "T"'] \arrow[d, "{[X,b]}"'] & {[TX,T^2B]} \arrow[r, "{[TX,\mu_B]}"'] \arrow[d, "{[TX,Tb]}"'] & {[TX,TB]} \arrow[dr, "{[TX,b]}"'] & \\
{[X,B]} \arrow[r, "T"'] & {[TX,TB]} \arrow[rr, "{[TX,b]}"'] & & {[TX,B]}
\end{tikzcd}$$

is a pseudo-map of algebras from $[X, \mathscr{B}]$ to $[TX, \mathscr{B}]$.

Proof. There are two coherence conditions for the two cell. That involving η follows directly from condition 4 of Proposition 8; that involving μ transforms by naturality to a consequence of condition 6 of Proposition 8. $\square$

We write $\sigma_{X,\mathscr{B}} = (\sigma_{X,\mathscr{B}}, \bar{\sigma}_{X,\mathscr{B}}) : [X, \mathscr{B}] \to [TX, \mathscr{B}]$ for the pseudo-map we have just constructed. We record some basic properties.

Proposition 13. *The following equalities hold in T-Alg.*

(i) $[\eta_X, \mathscr{B}] \cdot \sigma_{X,\mathscr{B}} = \mathrm{id}_{[X,\mathscr{B}]}$;
(ii) $\sigma_{TX,\mathscr{B}} \cdot \sigma_{X,\mathscr{B}} = [\mu_X, \mathscr{B}] \cdot \sigma_{X,\mathscr{B}}$.

Proof. The first follows directly from condition 5 of Proposition 8. The second transforms by naturality to a consequence of condition 7 of Proposition 8. $\square$

We now consider the naturality properties of $\sigma_{X,B}$. The first point is clear.

Proposition 14. *$\sigma_{X,B}$ is natural in X; that is, if $f : X \to Y$ in Cat, then $\sigma_{X,B} \cdot [f, B] = [Tf, B] \cdot \sigma_{Y,B}$.*

Equally clearly $\sigma_{X,B}$ is natural in $\mathscr{B}$ for strict maps of algebras, but it would be too much to expect naturality for pseudo-maps. Instead, we get pseudo-naturality.

Proposition 15. *Suppose that* $h : \mathscr{B} \to \mathscr{C}$ *is a pseudo-map of algebras. Then the* 2-*cell* $\sigma_{X,h}$ *defined by*

$$\begin{tikzcd}
{[X,B]} \arrow[r, "T"] \arrow[d, "{[X,h]}"'] & {[TX,TB]} \arrow[r, "{[TX,b]}"] \arrow[d, "{[TX,Th]}"'] \arrow[dr, phantom, "\overset{[TX,\bar{h}]}{\Rightarrow}"] & {[TX,B]} \arrow[d, "{[TX,h]}"] \\
{[X,C]} \arrow[r, "T"'] & {[TX,TC]} \arrow[r, "{[TX,c]}"'] & {[TX,C]}
\end{tikzcd}$$

is an invertible 2-*cell in* T-*Alg*, $\sigma_{X,h} : \sigma_{X,C} \cdot [X,h] \to [TX,h] \cdot \sigma_{TX,B}$. *Further more, the data* $\sigma_{X,h}$ *makes* $\sigma_{X,\mathscr{B}}$ *pseudo-natural in* $\mathscr{B}$.

Finally, we give an indication of how our strength axioms for γ are reflected in properties of σ. Two of these are straightforward.

Proposition 16. *We have the following correspondences.*

(i) *The* 1-*cell* $\sigma_{X,[Y,\mathscr{B}]} : [X,[Y,\mathscr{B}]] \to [TX,[Y,\mathscr{B}]]$ *is the exponential transpose of the* 1-*cell* $[t^*,\mathscr{B}] \cdot \sigma_{X\times Y,\mathscr{B}} : [X \times Y,\mathscr{B}] \to [TX \times Y,\mathscr{B}]$.
(ii) *The* 1-*cell* $[X,\sigma_{Y,\mathscr{B}}] : [X,[Y,\mathscr{B}]] \to [X,[TY,\mathscr{B}]]$ *is the exponential transpose of the* 1-*cell* $[t,\mathscr{B}] \cdot \sigma_{X\times Y,\mathscr{B}} : [X \times Y,\mathscr{B}] \to [X \times TY,\mathscr{B}]$.

Here (i) corresponds to condition 2 of Proposition 8 and (ii) corresponds to condition 3. As for condition 1, it is reflected in the following.

Proposition 17. *Pseudo-naturality of* σ *in* $\mathscr{B}$ *provides a canonical* 2-*cell*

$$\begin{tikzcd}
{[X,[Y,\mathscr{B}]]} \arrow[r, "\sigma_{X,[Y,\mathscr{B}]}"] \arrow[d, "{[X,\sigma_{Y,\mathscr{B}}]}"'] \arrow[dr, phantom, "\Rightarrow"] & {[TX,[Y,\mathscr{B}]]} \arrow[d, "{[TX,\sigma_{Y,\mathscr{B}}]}"] \\
{[X,[TY,\mathscr{B}]]} \arrow[r, "\sigma_{X,[TY,\mathscr{B}]}"'] & {[TX,[TY,\mathscr{B}]]}.
\end{tikzcd}$$

This is the exponential transpose of the 2-*cell* $[TX \times TY, b] \cdot [\gamma_{X,Y}, TB] \cdot T$.

5. 2-dimensional multilinear algebra

For pseudo-commutative T we wish to give data for T-*Alg* as a pseudo-closed 2-category, so we shall define for T-algebras $\mathscr{B}$ and $\mathscr{C}$ a function space T-algebra $[\mathscr{B},\mathscr{C}]$ and shall need to be able to characterise both pseudo- and strict maps of T-algebras from an arbitrary T-algebra $\mathscr{A}$ to it. These will correspond to what it is natural to call bilinear maps. At times we shall need to iterate this process: the T-algebra $\mathscr{C}$ will sometimes itself be of the form $[\mathscr{D},\mathscr{E}]$, and so on. So it is convenient to define a general notion of multilinear map, and to prove appropriate results connecting that with the various levels of iteration of the proposed pseudo-closed structure of T-*Alg*. (Though we are not considering enrichment in any kind of linear category, we maintain the terminology ‘multilinear map’ to signal that our subject is essentially a generalised linear algebra.)

The setting in which we describe our results on multilinear maps is that of 2-multicategories. The notions of multicategory and of symmetric multicategory make sense in very general contexts: in particular they make sense in the usual setting of enriched category theory. Thus we have a ready notion of (symmetric) 2- multicategory, and we shall describe such a structure on T-Alg.

5.1. The 2-multicategory of T-algebras

Before defining multilinear maps we treat a subsidiary topic. Suppose that X is a category and $\mathscr{A} = (A,a)$ and $\mathscr{B} = (B,b)$ are T-algebras. A *pseudo-map* $\mathscr{A} \to \mathscr{B}$ *parametrised by* X (or X-*indexed pseudo-map* $\mathscr{A} \to \mathscr{B}$) is given by a 1-cell $f : A \times X \to B$ and 2-cell $\bar{f}$ of shape

$$\begin{tikzcd}
TA \times X \arrow[r, "t^*"] \arrow[d, "Ta\times X"'] & T(A \times X) \arrow[r, "Tf"] \arrow[d, phantom, "\Downarrow \bar{f}"] & TB \arrow[d, "b"] \\
A \times X \arrow[rr, "f"'] & & B
\end{tikzcd}$$

satisfying easy extensions of the pseudo-map conditions. There is an obvious corresponding notion of 2-cell between parametrised pseudo-maps. We shall write T-$Alg(\mathscr{A},X;\mathscr{B})$ for the category of X-indexed pseudo-maps. We have obvious compositions

$$T\text{-}Alg(\mathscr{B},Y;\mathscr{C}) \times T\text{-}Alg(\mathscr{A},X;\mathscr{B}) \to T\text{-}Alg(\mathscr{A},X \times Y;\mathscr{C})$$

and actions

$$T\text{-}Alg(\mathscr{A},X;\mathscr{B}) \times Cat(Z,X) \to T\text{-}Alg(\mathscr{A},Z;\mathscr{B})$$

which taken all together form a natural structure. We do not spell this out. Note however that it makes sense to allow $\mathscr{A}$ as well as X to be absent in T-$Alg(\mathscr{A},X;\mathscr{B})$, we have the vacuous case T-$Alg(X;\mathscr{B}) = Cat(X,B)$. There is a clear connection between parametrised pseudo-maps and cotensors. We have

$$T\text{-}Alg(\mathscr{A},X;\mathscr{B}) \cong T\text{-}Alg(\mathscr{A}\colon [X,\mathscr{B}]).$$

By the symmetry in Cat we can just as easily parametrise maps on the right, or on both the left and right. So we have categories T-$Alg(X,\mathscr{A};\mathscr{B})$ and T-$Alg(X,\mathscr{A},Y;\mathscr{B})$ and we have well-behaved compositions also in these cases. Of course one is not gaining much: we have natural isomorphisms

$$\begin{aligned} T\text{-}Alg(X,\mathscr{A},Y;\mathscr{B}) &\cong T\text{-}Alg(X \times Y,\mathscr{A};\mathscr{B}) \cong T\text{-}Alg(X,\mathscr{A};[Y,\mathscr{B}]) \\ &\cong T\text{-}Alg(\mathscr{A},Y;[X,\mathscr{B}]) \cong T\text{-}Alg(Y \times X,\mathscr{A};\mathscr{B}). \end{aligned}$$

We draw attention to some natural examples of parametrised maps.

1. *Evaluation*. Let X be a category and $\mathscr{A}$ a T-algebra. Then there is an obvious evaluation map

$$\bullet : [X,\mathscr{A}] \times X \to \mathscr{A}.$$

This is a parametrised map of T-algebras. Under the natural isomorphism

$$T\text{-}Alg([X,\mathscr{A}],X;\mathscr{A}) \cong T\text{-}Alg([X,\mathscr{A}],[X,\mathscr{A}])$$

it of course corresponds to the identity on $[X,\mathscr{A}]$.

2. *Application*. Let $\mathscr{A}$ and $\mathscr{B}$ be T-algebras. Then there is an obvious action

$$\bullet : T\text{-}Alg(\mathscr{A},\mathscr{B}) \times \mathscr{A} \to \mathscr{B}$$

of the category of pseudo-maps: this is a parametrised map of T-algebras.

3. *The unit*. In $T\text{-}Alg(X,FX) = Cat(X,TX)$ we have $\eta_X : X \to TX$ which thus can be regarded as a (vacuous) parametrised map. Composing with it in the sense explained above gives a map

$$T\text{-}Alg(FX,\mathscr{A}) \to T\text{-}Alg(X;\mathscr{A}) = Cat(X,A)$$

which we easily identify with the association $\bar{e}_{X,\mathscr{A}}$ from the biadjunction which we gave in Section 4.

In Section 3.5 we observed that the strength gives a unique map

$$t_i : A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_n \to T(A_1 \times \cdots \times A_n);$$

we use these in our definition of multilinear map.

Definition 10. A *multilinear* map of T-algebras from $(A_1,a_1),\ldots,(A_n,a_n)$ to (B,b) consists of

- a 1-cell $h : A_1 \times \cdots \times A_n \to B$
- for each i, a 2-cell

$$\begin{tikzcd}
A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_n \arrow[r, "t_i"] \arrow[d, dash] & T(A_1 \times \cdots \times A_n) \arrow[r, "Th"] & TB \arrow[dd, "b"] \\
A_1 \times \cdots \times A_{i-1} \times a_i \times A_{i+1} \times \cdots \times A_n \arrow[d] & \Downarrow \bar{h}_i & \\
A_1 \times \cdots A_n \arrow[rr, "h"'] & & B
\end{tikzcd}$$

such that, for each i, $(h,\bar{h}_i)$ is a parametrised map of T-algebras (in the sense sketched above), and if i is less than j, the evident two pastings from

$$A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_{j-1} \times TA_j \times A_{j+1} \times \cdots \times A_n$$

to B, expressing the idea that h_i and h_j commute with each other in a 2-dimensional sense, are equal. We say that the multilinear map $h = (h,\bar{h}_i)$ is *strict in i* just when the 2-cell $\bar{h}_i$ is the identity.

We should be precise about the commutativity axiom. In order to avoid clutter, we shall write the axiom explicitly for the case of a bilinear map; and then we shall explain the routine extension to the multilinear case. One requires that the diagram

$$\begin{tikzcd}
TA\times TB \arrow[r, "t^*"] \arrow[dd, "TA\times b"'] & T(A\times TB) \arrow[r, "T"] \arrow[dd, "T(A\times b)"'] & T^2(A\times B) \arrow[r, "\mu"] \arrow[d, "T^2h"] & T(A\times B) \arrow[d, "Th"] \\
 & {} \arrow[r, phantom, "\Downarrow T\bar{h}_A"] & T^2C \arrow[r, "\mu"'] \arrow[d, "Tc"] & TC \arrow[dd, "c"] \\
TA\times B \arrow[r, "t^*"] \arrow[d, "a\times B"'] & T(A\times B) \arrow[r, "Th"'] \arrow[d, phantom, "\Downarrow\bar{h}_B"] & TC \arrow[dr, "c"] & \\
A\times B \arrow[rrr, "h"'] & {} & & C
\end{tikzcd}$$

is the result of pasting $\gamma_{A,B}$ on top of the diagram

$$\begin{tikzcd}
TA\times TB \arrow[r, "t"] \arrow[dd, "a\times TB"'] & T(TA\times B) \arrow[r, "Tt^*"] \arrow[dd, "T(a\times B)"'] & T^2(A\times B) \arrow[r, "\mu"] \arrow[d, "T^2h"] & T(A\times B) \arrow[d, "Th"] \\
 & {} \arrow[r, phantom, "\Downarrow T\bar{h}_B"] & T^2C \arrow[r, "\mu"'] \arrow[d, "Tc"] & TC \arrow[dd, "c"] \\
A\times TB \arrow[r, "t"] \arrow[d, "A\times b"'] & T(A\times B) \arrow[r, "Th"'] \arrow[d, phantom, "\Downarrow\bar{h}_A"] & TC \arrow[dr, "c"] & \\
A\times B \arrow[rrr, "h"'] & {} & & C
\end{tikzcd}$$

One should think that the two diagrams are equal modulo γ. In the general case, we have diagrams with top edges equal to the boundary of

$$\begin{tikzcd}
(\cdots TA_i\times\cdots\times TA_j\cdots) \arrow[r, "t_i"] \arrow[d, "t_j"'] & T(A_1\times\cdots TA_j\cdots\times A_n) \arrow[r, "Tt_j"] & T^2(A_1\times\cdots\times A_n) \arrow[d, "\mu"] \\
T(A_1\times\cdots TA_i\cdots\times A_n) \arrow[r, "Tt_i"'] & T^2(A_1\times\cdots\times A_n) \arrow[r, "\mu"'] & T(A_1\times\cdots\times A_n)
\end{tikzcd}$$

By Proposition 5 there is a unique 2-cell $\gamma_{i,j}$ constructed from positive versions of γ which fills the hole. We require that our two pastings of $\bar{h}_i$ and $\bar{h}_j$ are equal modulo $\gamma_{i,j}$.

The notion of a 2-cell between pseudo-maps of T-algebras extends easily to a notion of 2-*cell between multilinear maps*. A 2-cell from $(h,\bar{h}_i)$ to $(k,\bar{k}_i)$ is a 2-cell $\rho : h \Rightarrow k$ which satisfies

$$\bar{k}_i \cdot (T\rho \cdot t_i) = \rho \cdot \bar{h}_i$$

for all i. We write $T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;\mathscr{B})$ for the category of multilinear maps from $\mathscr{A}_1,\ldots,\mathscr{A}_n$ to $\mathscr{B}$.

It is clear how to compose multilinear maps. Given

$$(f,\bar{f}_i) : \mathscr{A}_1 \times \cdots \times \mathscr{A}_n \to \mathscr{B}_r$$

and

$$(g,\bar{g}_j) : \mathscr{B}_1 \times \cdots \times \mathscr{B}_m \to \mathscr{C},$$

we compose f and g at r to get

$$(h,\bar{h}_{A_i},\bar{h}_{B_j})) : \mathscr{B}_1 \times \cdots \mathscr{A}_1 \times \cdots \times \mathscr{A}_n \cdots \times \mathscr{B}_m \to \mathscr{B}_r,$$

where we set $h = g \cdot (\cdots B_{r-1} \times f \times B_{r+1} \cdots)$, $\bar{h}_{A_i} = \bar{g}_r \cdot (\cdots B_{r-1} \times \bar{f}_i \times B_{r+1} \cdots)$ and $\bar{h}_{B_j} = \bar{g}_j \cdot (\cdots B_{r-1} \times f \times B_{r+1} \cdots)$. Checking that this composite is indeed a multilinear map is essentially routine, though it is of course reliant on the uniqueness of the 2-cells $\gamma_{i,j}$ from Proposition 5. The definition of composition clearly extends to 2-cells between multilinear maps, and we have the following.

Proposition 18. *Suppose that T is a pseudo-commutative* 2-*monad. The structure T-Alg consisting of T-algebras, multilinear maps of T-algebras and* 2 *cells between multilinear maps, together with the evident identities and compositions forms a* 2-*multicategory. If the pseudo-commutativity γ is symmetric, then T-Alg is a symmetric* 2-*multicategory.*

Proof. This involves routine checking. Again, we rely on Proposition 5 in checking the associativity of composition. We use the symmetry of the commutativity γ to show that the condition that the $\bar{h}_i$ commute with each other is preserved under the action of the symmetric groups. □

Note that the 2-multicategory T-*Alg* extends the 2-category T-*Alg* in the obvious sense. For $n = 1$ an n-multimap is just a pseudo-map of T-algebras; and it is strict as a multimap just if it is a strict map.

We started this section with a notion of parametrised pseudo-map. We can rerun the idea to give parametrisation of multilinear maps. We get categories $T\text{-}Alg(X,\mathscr{A}_1,\ldots,\mathscr{A}_n;\mathscr{B})$, $T\text{-}Alg(\mathscr{A}_1,X,\ldots,\mathscr{A}_n;\mathscr{B}),\ldots,T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n,X;\mathscr{B})$ of multimaps from $\mathscr{A}_1,\ldots,\mathscr{A}_n$ to $\mathscr{B}$ parametrised by X. (They are isomorphic: it does not matter where the X appears.) Again, there are obvious compositions and actions which form a natural structure which we do not discuss here.

5.2. Forgetful and free functors

Note that Cat is a 2-multicategory since it is a 2-category with products. Then the forgetful functor $U : T\text{-}Alg \to Cat$ is in the obvious sense a map of 2-multicategories. Generally we have

$$T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;\mathscr{B}) \xrightarrow{U} Cat(A_1 \times \cdots \times A_n, B).$$

In case $n=1$ this agrees with the ordinary 2-categorical U, while in case $n=0$ we get the identity

$$T\text{-}Alg((\,);\mathscr{B}) = Cat(1,B) \xrightarrow{U} Cat(1,B).$$

Now in the context of 2-multicategories we can also consider the partial effect of the forgetful functor. For example, we have

$$U_1 : T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;\mathscr{B}) \to T\text{-}Alg(A_1,\mathscr{A}_2,\ldots,\mathscr{A}_n;\mathscr{B})$$

forgetting just the algebra structure on $\mathscr{A}_1$ and the data related to it. Similarly we have, for all appropriate i, functors U_i taking categories of multilinear maps to categories of *parametrised multilinear maps* in the obvious extension of the notion. The U_i respect composition in the sense that, for example,

$$\begin{tikzcd}
T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;\mathscr{B})\times T\text{-}Alg(\mathscr{B},\mathscr{B}_1,\ldots,\mathscr{B}_m;\mathscr{C}) \arrow[r, "comp"] \arrow[d, "U_1\times \mathrm{id}"'] & T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{B}_m;\mathscr{C}) \arrow[d, "U_1"] \\
T\text{-}Alg(A_1,\ldots,\mathscr{A}_n;\mathscr{B})\times T\text{-}Alg(\mathscr{B},\mathscr{B}_1,\ldots,\mathscr{B}_m;\mathscr{C}) \arrow[r, "comp"] & T\text{-}Alg(A_1,\ldots,\mathscr{B}_m;\mathscr{C})
\end{tikzcd}$$

commutes. Furthermore we can use the free functor F to go back from parametrised multilinear maps to multilinear maps. For example we have a functor

$$F_1 : T\text{-}Alg(X,\mathscr{A}_2,\ldots,\mathscr{A}_n;\mathscr{B}) \to T\text{-}Alg(FX,\mathscr{A}_2,\ldots,\mathscr{A}_n;\mathscr{B})$$

taking an X-parametrised map to a multilinear map strict in FX. The F_i similarly respect composition. The biadjunction $F \dashv U$ is reflected in the fact that the composite

$$T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;\mathscr{B}) \to T\text{-}Alg(A_1,\ldots,\mathscr{A}_n;\mathscr{B}) \to T\text{-}Alg(FA_1,\ldots,\mathscr{A}_n;\mathscr{B})$$

is induced by the counit ε, while the composite

$$T\text{-}Alg(X,\ldots,\mathscr{A}_n;\mathscr{B}) \to T\text{-}Alg(FX,\ldots,\mathscr{A}_n;\mathscr{B}) \to T\text{-}Alg(TX,\ldots,\mathscr{A}_n;\mathscr{B})$$

admits a retract induced by η.

5.3. Multilinear universality

We need to use a notion of limit in a 2-multicategory which is a trivial generalisation of the notion of weighted limit. Suppose that $\mathscr{D}$ is a (small) 2-category and $F:\mathscr{D}\to Cat$ a 2-functor. If $\mathscr{K}$ is a 2-category, then an F-weighted cone over $G:\mathscr{D}\to\mathscr{K}$ with vertex A is an object of the category

$$[\mathscr{D}, Cat](F, \mathscr{K}(A, G(-)));$$

and an F-weighted limit is a representing object $\lim(F,G)$, that is an isomorphism of categories

$$\mathscr{K}(A, \lim(F,G)) \cong [\mathscr{D}, Cat](F, \mathscr{K}(A, G(-))),$$

natural in A. Now suppose that $\mathscr{K}$ is a 2-multicategory. We extend the notion of weighted cone and limit as follows. An F-weighted cone with vertices $A_1,\ldots,A_n$ is an object of

$$[\mathscr{D}, Cat](F, \mathscr{K}(A_1,\ldots,A_n; G(-)))$$

and an F-weighted limit is a representing object $\lim(F,G)$, that is an isomorphism of categories

$$\mathscr{K}(A_1,\ldots,A_n; \lim(F,G)) \cong [\mathscr{D}, Cat](F, \mathscr{K}(A_1,\ldots,A_n; G(-))),$$

natural in the obvious multicategorical sense in $A_1,\ldots,A_n$.

As we explained for T pseudo-commutative, T-Alg is not just a 2-category but a 2-multicategory. The arguments of [3] extend readily to this situation and we have the following.

Proposition 19. *T-Alg has PIE limits as a 2-multicategory.*

6. Pseudo-closed and pseudo-monoidal structure

In this section we take a pseudo-commutative 2-monad T on Cat and exhibit a pseudo-closed structure on the 2-category T-Alg. When our pseudo-commutativity is symmetric we find that we have a symmetric pseudo-closed structure. We first show that for any T-algebras $\mathscr{A}$ and $\mathscr{B}$ the category T-$Alg(\mathscr{A},\mathscr{B})$ has a T-algebra structure defined pointwise; that is, it inherits a T-algebra structure from the cotensor $[A,\mathscr{B}]$. This exponential or function space has properties one expects with respect to the multilinear maps in T-Alg, and we use this first to define the data for a closed structure on the 2-category and then to verify the axioms. We close by explaining how it then follows that we have a pseudo-monoidal structure on T-Alg, which is thus pseudo-monoidal closed.

6.1. The T-algebra of pseudo-maps

In order to provide a pseudo-closed structure on T-Alg, we emulate the proof that for an ordinary commutative monad on Set, the category of algebras is closed. For an ordinary commutative monad T, the closed structure is given by internalizing the notion of the collection of structure preserving maps between algebras. Over *Set* it is clear that commutativity of T gives this collection the structure of a T-algebra defined pointwise. In the general abstract setting of a strong monad on a commutative monoidal category, one expresses the closed structure using an equaliser: simple categorical arguments show that for T commutative, the equaliser inherits the pointwise T-algebra structure [12]. By analogy we wish to internalize the notion of the category of structure preserving maps in the pseudo sense. Over *Cat* we can just take the category of pseudo-maps of algebras, but it is tiresome to show with bare hands that this has a T-algebra structure defined pointwise. It is more elegant to express the category $T\text{-}Alg(\mathscr{A},\mathscr{B})$ of pseudo-maps from $\mathscr{A}$ to $\mathscr{B}$ as a limit built from an iso-inserter and two equifiers in *Cat*; the limit diagrams are diagrams in $T\text{-}Alg$, and so the limit lifts to $T\text{-}Alg$. In view of [3], we might as well work in $T\text{-}Alg$ straight away.

Definition 11. Given T-algebras $\mathscr{A} = (A, a)$ and $\mathscr{B} = (B, b)$, we construct a new T-algebra in three steps.

1. Take the iso-inserter $(i : In \to [A,\mathscr{B}], \alpha')$ of
$$\begin{tikzcd}[A,\mathscr{B}] \arrow[r, shift left, "\sigma_{A,\mathscr{B}}"] \arrow[r, shift right, "{[a,\mathscr{B}]}"'] & {[TA,\mathscr{B}]}\end{tikzcd}$$
So we get a universal 2-cell $\alpha' : \sigma_{A,\mathscr{B}} \cdot i \to [a,\mathscr{B}] \cdot i$. Note that we use conditions from Proposition 8 in showing that the above is a diagram in $T\text{-}Alg$.
2. Take the equifier $e' : Eq' \to In$ of $[\eta_A,\mathscr{B}] \cdot \alpha'$ with the identity. Note that by (i) of Proposition 13 this makes sense.
3. Take the equifier $e : Eq \to Eq'$ of $[\mu_A, B] \cdot \alpha' \cdot e'$ with the following pasting:

$$\begin{tikzcd}
 & & {[A,\mathscr{B}]} \arrow[r, "\sigma"] & {[TA,\mathscr{B}]} \arrow[dr, "\sigma"] & \\
Eq' \arrow[r, "e'"] & In \arrow[ur, "i"] \arrow[r, "i"] \arrow[dr, "i"'] & {[A,\mathscr{B}]} \arrow[u, phantom, "\Downarrow\alpha'"] \arrow[d, phantom, "\Downarrow\alpha'"] \arrow[ur, "{[a,B]}"'] \arrow[dr, "\sigma"] & & {[T^2A,\mathscr{B}]} \\
 & & {[A,\mathscr{B}]} \arrow[r, "{[a,B]}"'] & {[TA,\mathscr{B}]} \arrow[ur, "{[Ta,B]}"'] &
\end{tikzcd}$$

Here the final square commutes by the easy naturality of σ, and the domains of the 2-cells match easily; for the codomains we use (ii) of Proposition 13.

We write the resulting T-algebra $[\mathscr{A},\mathscr{B}]$ and call it, equipped with the composite

$$p = i \cdot e' \cdot e : [\mathscr{A},\mathscr{B}] \to [A,\mathscr{B}]$$

and the isomorphic 2-cell

$$\alpha = \alpha' \cdot e' \cdot e : \sigma_{A,\mathscr{B}} \cdot p \to [a,\mathscr{B}] \cdot p$$

the function space $\mathscr{A}$ to $\mathscr{B}$.

If in *Cat* we take the canonical notions of iso-inserter and equifier we shall find that our final *Eq* is exactly the category of pseudo-maps from $\mathscr{A}$ to $\mathscr{B}$. So the forgetful 2-functor takes $[\mathscr{A},\mathscr{B}]$ to $T\text{-}Alg(\mathscr{A},\mathscr{B})$. Moreover the following universal property follows directly from the construction.

Proposition 20. *Suppose given a pair of T-algebras $\mathscr{A}=(A,a)$ and $\mathscr{B}=(B,b)$.*

(i) *The T-algebra $[\mathscr{A},\mathscr{B}]$ equipped with*

$$p:[\mathscr{A},\mathscr{B}]\to[A,\mathscr{B}] \text{ and an isomorphic 2-cell } \alpha:\sigma_{A,\mathscr{B}}\cdot p\to[a,\mathscr{B}]\cdot p$$

satisfies the universal property: for each $\mathscr{D}$, composition with p induces an isomorphism between $T\text{-}Alg(\mathscr{D},[\mathscr{A},\mathscr{B}])$ and the category of cones given by data

$$f:\mathscr{D}\to[A,\mathscr{B}] \text{ and an isomorphic 2-cell } \beta:\sigma_{A,\mathscr{B}}\cdot f\to[a,\mathscr{B}]\cdot f$$

satisfying the two equification conditions:

$$\begin{tikzcd}
& {[A,\mathscr{B}]} \arrow[dr, "\sigma"] & & \\
\mathscr{D} \arrow[ur, "f"] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\beta"] & & {[TA,\mathscr{B}]} \arrow[r, "\eta"] & {[A,\mathscr{B}]} = \mathrm{id}_f \\
& {[A,\mathscr{B}]} \arrow[ur, "{[a,\mathscr{B}]}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& {[A,\mathscr{B}]} \arrow[dr, "\sigma"] & & \\
\mathscr{D} \arrow[ur, "f"] \arrow[dr, "f"'] \arrow[rr, phantom, "\Downarrow\beta"] & & {[TA,\mathscr{B}]} \arrow[r, "\mu"] & {[T^2A,\mathscr{B}]} \\
& {[A,\mathscr{B}]} \arrow[ur, "{[a,\mathscr{B}]}"'] & &
\end{tikzcd}
=
\begin{tikzcd}
& {[A,\mathscr{B}]} \arrow[rr, "\sigma"] & & {[TA,\mathscr{B}]} \arrow[dr, "\sigma"] & \\
\mathscr{D} \arrow[ur, "f"] \arrow[r, "f"] \arrow[dr, "f"'] & {[A,\mathscr{B}]} \arrow[urr, "{[a,B]}"] \arrow[drr, "\sigma"] \arrow[u, phantom, "\Downarrow\beta"] \arrow[d, phantom, "\Downarrow\beta"] & & & {[T^2A,\mathscr{B}]} \\
& {[A,\mathscr{B}]} \arrow[rr, "{[a,B]}"'] & & {[TA,\mathscr{B}]} \arrow[ur, "{[Ta,B]}"'] &
\end{tikzcd}$$

By Section 5.3 we can take this in the sense of 2-multicategories: composition with k gives an isomorphism between the category $T\text{-}Alg(\mathscr{C}_1,\ldots,\mathscr{C}_n;[\mathscr{A},\mathscr{B}])$ and the category of cones with vertices $\mathscr{C}_1,\ldots,\mathscr{C}_n$.

(ii) *k is a strict map of T-algebras and postcomposition with k reflects strictness.*
(iii) *The forgetful functor $T\text{-}Alg \to Cat$ takes*

$$[\mathscr{A},\mathscr{B}] \text{ with } k : [\mathscr{A},\mathscr{B}] \to [A,\mathscr{B}] \text{ and } \alpha : \sigma_{A,\mathscr{B}} \cdot k \to [a,\mathscr{B}] \cdot k$$

to corresponding limit data

$$T\text{-}Alg(\mathscr{A},\mathscr{B}) \text{ with } k : T\text{-}Alg(\mathscr{A},\mathscr{B}) \to [A,B] \text{ and } \alpha : \sigma_{A,\mathscr{B}} \cdot k \to [a,B] \cdot k.$$

We deduce by a routine use of the universal property of $[\mathscr{A},\mathscr{B}]$ the following.

Theorem 9. *$[-,-]$ extends to a 2-functor from $T\text{-}Alg^{op} \times T\text{-}Alg$ to $T\text{-}Alg$.*

Finally, we recall the adjoint retract equivalence $\bar{i}_{X,\mathscr{A}} \dashv \bar{e}_{X,\mathscr{A}}$ from 4.1. By Proposition 20 the retraction

$$\bar{e} = \bar{e}_{X,\mathscr{A}} = T\text{-}Alg(FX,\mathscr{A}) \xrightarrow{U} Cat(TX,A) \xrightarrow{Cat(\eta,\ A)} Cat(X,A)$$

lifts to a strict map

$$e = e_{X,\mathscr{A}} = [FX,\mathscr{A}] \xrightarrow{p} [TX,\mathscr{A}] \xrightarrow{[\eta,A]} [X,\mathscr{A}]$$

in $T\text{-}Alg$; we have $U(e_{X,\mathscr{A}}) = \bar{e}_{X,\mathscr{A}}$. Now the forgetful $U : T\text{-}Alg \to Cat$ reflects equivalences, and more specifically adjoint retract equivalences. The naturality properties also lift so we have the following.

Theorem 10. *The (retract) equivalence $\bar{i}_{X,\mathscr{A}} \dashv \bar{e}_{X,\mathscr{A}}$ lifts to an equivalence $i_{X,\mathscr{A}} \dashv e_{X,\mathscr{A}}$.*
$e_{X,\mathscr{A}}$ is natural in X and $\mathscr{A}$; $i_{X,\mathscr{A}}$ is natural in X and pseudo-natural in $\mathscr{A}$.

6.2. Multilinear properties of the exponential

Theorem 11. *Let $\mathscr{A}_1,\ldots,\mathscr{A}_n$, $\mathscr{B}$ and $\mathscr{C}$ be T-algebras. Exponentiation induces a natural isomorphism*

$$T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n,\mathscr{B};\mathscr{C}) \cong T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;[\mathscr{B},\mathscr{C}])$$

between the indicated categories of multilinear maps. Moreover, the map into $[\mathscr{B},\mathscr{C}]$ is strict in i if and only if the corresponding map to $\mathscr{C}$ is strict in i; and it factors through the category of strict maps of T-algebras if and only if the corresponding map is strict in $\mathscr{B}$.

Proof. The proof of this is largely routine, and given the rest, the points about strictness are obvious. To keep things simple, we first do the simplest case: we show

$$T\text{-}Alg(\mathscr{A},\mathscr{B};\mathscr{C}) \cong T\text{-}Alg(\mathscr{A},[\mathscr{B},\mathscr{C}]).$$

An object of the left-hand side is given by data $(f, \bar{f}_A, \bar{f}_B)$ where

$$f : A \times B \to C,$$

$$\bar{f}_{A:\mathscr{C}} \cdot Tf \cdot t^* \to f \cdot (a \times B),$$

$$\bar{f}_{B:\mathscr{C}} \cdot Tf \cdot t \to f \cdot (A \times b)$$

satisfy two pseudo-map conditions and a commutativity condition.

The exponential transpose of f is a 1-cell $g : A \to [B, C]$, and the pseudo-map condition for $\bar{f}_A$ says exactly that the transpose of $\bar{f}_A$ gives a pseudo-map of algebras $(g, \bar{g}) : \mathscr{A} \to [B, \mathscr{C}]$. Transposing $\bar{f}_B$ gives an invertible 2-cell

$$\begin{tikzcd} A \arrow[r, shift left=2, "\sigma \cdot g"] \arrow[r, shift right=2, "{[b,C] \cdot g}"'] \arrow[r, phantom, "\Downarrow_\alpha"] & {[TB, \mathscr{C}]} \end{tikzcd}$$

in *Cat*. The commutativity condition says exactly that this lifts to a 2-cell

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left=2, "\sigma \cdot g"] \arrow[r, shift right=2, "{[b,C] \cdot g}"'] \arrow[r, phantom, "\Downarrow_\alpha"] & {[TB, \mathscr{C}]} \end{tikzcd}$$

in *T-Alg*. Finally the pseudo-map condition for $\bar{f}_B$ gives exactly the equifying conditions. Hence by universality the data above corresponds exactly to a pseudo-map from $\mathscr{A}$ to $[\mathscr{B}, \mathscr{C}]$, i.e. to an object of the right-hand side. All these correspondences are natural, so we obtain not just a bijection on objects but an isomorphism of categories natural in the data. We now illustrate how to extend this to arbitrary multimaps. Suppose we wish to show

$$T\text{-}Alg(\mathscr{A}, \mathscr{B}, \mathscr{C}; \mathscr{D}) \cong T\text{-}Alg(\mathscr{A}, \mathscr{B}; [\mathscr{C}, \mathscr{D}]).$$

An object of the left-hand side is now given by data $h, \bar{h}_A, \bar{h}_B$ and $\bar{h}_C$, satisfying three pseudo-map conditions and three commutativity conditions. Transposing h gives a 1-cell $A \times B \to [C, D]$; and the transposes of $\bar{h}_A$ and $\bar{h}_B$ give the data for a multilinear map $\mathscr{A} \times \mathscr{B} \to [C, \mathscr{D}]$; that it is such uses one commutativity condition. The 2-cell $\bar{h}_C$ transposes to give an invertible 2-cell

$$\begin{tikzcd} A \times B \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow"] & {[TC, D]} \end{tikzcd}$$

in *Cat*, and the two further commutativity conditions involving $\bar{h}_C$ say exactly that this lifts to

$$\begin{tikzcd} \mathscr{A} \times \mathscr{B} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow"] & {[TC, \mathscr{D}]} \end{tikzcd}$$

in *T-Alg*. Finally, the pseudo-map condition for $\bar{h}_C$ gives the equifying conditions of the limit. So by multilinear universality the data corresponds exactly to an object of the right-hand side. This is all routine, but one should note the tacit uses of Proposition 5. □

We have occasional need of some simple extensions of Theorem 11. Clearly, the argument extends to categories of parametrised maps of the kind introduced in Section 5. Moreover, the partial actions of U and F described there respect exponential transpose. We restrict the formulation to avoid notational fuss.

Proposition 21. *Exponentiation induces natural isomorphisms as indicated in the following commutative diagrams.*

$$\begin{tikzcd} T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n,\mathscr{B};\mathscr{C}) \arrow[r, "\cong"] \arrow[d, "U_1"'] & T\text{-}Alg(\mathscr{A}_1,\ldots,\mathscr{A}_n;[\mathscr{B},\mathscr{C}]) \arrow[d, "U_1"] \\ T\text{-}Alg(A_1,\ldots,\mathscr{A}_n,\mathscr{B};\mathscr{C}) \arrow[r, "\cong"] & T\text{-}Alg(A_1,\ldots,\mathscr{A}_n;[\mathscr{B},\mathscr{C}]) \end{tikzcd}$$

$$\begin{tikzcd} T\text{-}Alg(X,\ldots,\mathscr{A}_n,\mathscr{B};\mathscr{C}) \arrow[r, "\cong"] \arrow[d, "F_1"'] & T\text{-}Alg(X,\ldots,\mathscr{A}_n;[\mathscr{B},\mathscr{C}]) \arrow[d, "F_1"] \\ T\text{-}Alg(FX,\ldots,\mathscr{A}_n,\mathscr{B};\mathscr{C}) \arrow[r, "\cong"] & T\text{-}Alg(FX,\ldots,\mathscr{A}_n;[\mathscr{B},\mathscr{C}]) \end{tikzcd}$$

6.3. Exponential transpose

We now give some examples of maps arising by exponential transpose.

1. *Evaluation*. We need notation for the canonical evaluation map. We write

$$ev = (ev, \overline{ev}) : [\mathscr{A},\mathscr{B}] \times \mathscr{A} \to \mathscr{B}$$

for the map corresponding to the identity $[\mathscr{A},\mathscr{B}] \to [\mathscr{A},\mathscr{B}]$. This is strict in $[\mathscr{A},\mathscr{B}]$ and the 2-cell $\overline{ev}$ is the transpose of the 2-cell α arising in the definition of $[\mathscr{A},\mathscr{B}]$.

2. *Composition*. We also make use of an internalisation of composition. We can compose two evaluations

$$[\mathscr{A},\mathscr{B}] \times \mathscr{A} \to \mathscr{B} \quad \text{and} \quad [\mathscr{B},\mathscr{C}] \times \mathscr{B} \to \mathscr{C}$$

to give a multilinear map which we write with evident notation

$$ev \cdot (1 \times ev) : [\mathscr{B},\mathscr{C}] \times [\mathscr{A},\mathscr{B}] \times \mathscr{A} \to [\mathscr{B},\mathscr{C}] \times B \to \mathscr{C};$$

we define the exponential transpose of this to be

$$comp : [\mathscr{B},\mathscr{C}] \times [\mathscr{A},\mathscr{B}] \to [\mathscr{A},\mathscr{C}].$$

We make precise a sense in which *comp* internalises composition. First consider $U_1(ev \cdot (1 \times ev))$. Since U_1 respects composition, this is the composite

$$U_1(ev \cdot (1 \times ev)) : T\text{-}Alg(\mathscr{B},\mathscr{C}) \times [\mathscr{A},\mathscr{B}] \times \mathscr{A} \to T\text{-}Alg(\mathscr{B},\mathscr{C}) \times B \to \mathscr{C}.$$

By naturality of *ev* we can rewrite that as

$$U_1(ev \cdot (1 \times ev)) : T\text{-}Alg(\mathscr{B},\mathscr{C}) \times [\mathscr{A},\mathscr{B}] \times \mathscr{A} \to [\mathscr{A},\mathscr{C}] \times A \to \mathscr{C}.$$

Now since U_1 respects exponential transpose we get that

$$U_1(comp) : T\text{-}Alg(\mathscr{B},\mathscr{C}) \times [\mathscr{A},\mathscr{B}] \to [\mathscr{A},\mathscr{C}]$$

is just the functoriality action.

3. *Functoriality*. Recall from Section 5 the action $\bullet : T\text{-}Alg(\mathscr{A},\mathscr{B}) \times \mathscr{A} \to \mathscr{B}$. We consider its exponential transpose. Under the natural isomorphism

$$\begin{aligned} T\text{-}Alg(T\text{-}Alg(\mathscr{A},\mathscr{B}),\mathscr{A};\mathscr{B}) &\cong T\text{-}Alg(T\text{-}Alg(\mathscr{A},\mathscr{B});[\mathscr{A},\mathscr{B}]) \\ &= Cat(T\text{-}Alg(\mathscr{A},\mathscr{B}),T\text{-}Alg(\mathscr{A},\mathscr{B})), \end{aligned}$$

it of course corresponds to the identity on $T\text{-}Alg(\mathscr{A},\mathscr{B})$. We note that we can now give further parametrised maps. For example, as $[\mathscr{A},\mathscr{B}]$ is functorial in $\mathscr{A}$, we have a map $T\text{-}Alg(\mathscr{B},\mathscr{C}) \to T\text{-}Alg([\mathscr{A},\mathscr{B}],[\mathscr{A},\mathscr{C}])$. Composing that with a version of the $\bullet$ we just considered gives a parametrised map

$$\bullet : T\text{-}Alg(\mathscr{B},\mathscr{C}) \times [\mathscr{A},\mathscr{B}] \to [\mathscr{A},\mathscr{C}],$$

which is again the functoriality action.

4. *The section $i_{X,\mathscr{A}}$* We already obtained $i_{X,\mathscr{A}} : [X,\mathscr{A}] \to [FX,\mathscr{A}]$ by lifting the retract equivalence $\bar{i}_{X,\mathscr{A}} \dashv \bar{e}_{X,\mathscr{A}}$. Following through the proof of Theorem 11 enables us to identify the exponential transpose

$$[X,\mathscr{A}] \times FX \to \mathscr{A}.$$

It is given precisely by the action of

$$F_2 : T\text{-}Alg([X,\mathscr{A}],X;\mathscr{A}) \to T\text{-}Alg([X,\mathscr{A}],FX;\mathscr{A})$$

on the canonical evaluation

$$\bullet : [X,\mathscr{A}] \times X \to \mathscr{A}.$$

5. *The retraction $e_{X,\mathscr{A}}$* We constructed maps $e_{X,\mathscr{C}} : [FX,\mathscr{C}] \to [X,\mathscr{C}]$ lifting the biadjunction correspondence $\bar{e}_{X,\mathscr{C}}$. These induce functors

$$T\text{-}Alg(\mathscr{A},[FX,\mathscr{C}]) \to T\text{-}Alg(\mathscr{A},[X,\mathscr{C}])$$

and hence functors

$$T\text{-}Alg(\mathscr{A},FX;\mathscr{C}) \to T\text{-}Alg(\mathscr{A},X;\mathscr{C}),$$

which can be regarded as parametrised versions of $\bar{e}$. We read off a description from the proof of Theorem 11 and deduce that these functors are induced by composition with the vacuous parametrised map $X \to FX$.

6.4. Pseudo-closed structure on T-Alg

We are now in a position to present the data to exhibit T-Alg as a pseudo-closed 2-category and prove that the data satisfies the axioms of Section 2.1.

The data: T-Alg is a 2-category and we take $V: T\text{-}Alg \to Cat$ to be the forgetful functor $U: T\text{-}Alg \to Cat$. (We shall stick with U in what follows.) We saw from its universal construction that the internal horn $[-,-]: T\text{-}Alg^{op} \times T\text{-}Alg \to Cat$ is 2-functorial. To describe the rest of the structure we make use of the 2-multicategorical structure of T-Alg together with the canonical correspondences $\bar{e}$ and $\bar{i}$ of the biadjunction $F \dashv U$ and their lifts e and i.

The unit: We take as unit, $F1$, the free T-algebra on 1. (Note that strict maps $F1 \to \mathscr{A}$ in T-Alg correspond to objects of the underlying category A.)

Identity: The 1-cell $j: F1 \to [\mathscr{A}, \mathscr{A}]$ is defined to be the strict map corresponding under the adjunction to the functor $1 \to T\text{-}Alg(\mathscr{A}, \mathscr{A})$ picking out the identity on $\mathscr{A}$. Write $\hat{j}_{\mathscr{A}}: 1 \to T\text{-}Alg(\mathscr{A}, \mathscr{A})$ for this 1-cell in Cat. Then in terms of the retract equivalence $\bar{i} \dashv \bar{e}$ we have effectively defined $j_{\mathscr{A}} = \bar{i}_{1,[\mathscr{A},\mathscr{A}]}(\hat{j}_{\mathscr{A}})$ or equivalently as the unique strict map with $\bar{e}_{1,[\mathscr{A},\mathscr{A}]}(j_{\mathscr{A}}) = \hat{j}_{\mathscr{A}}$.

(We note in passing that we can describe j as the transpose of a bilinear map with underlying 1-cell

$$T1 \times A \xrightarrow{t^*} T(1 \times A) \xrightarrow{Tl} TA \xrightarrow{a} A,$$

which is strict in $T1$ and with an A-component 2-cell which we omit.)

Unit laws: $e_{\mathscr{A}}: [F1, \mathscr{A}] \longrightarrow \mathscr{A}$ is the composite of $e_{1,\mathscr{A}}: [F1, \mathscr{A}] \longrightarrow [1, \mathscr{A}]$ and the isomorphism $[1, \mathscr{A}] \longrightarrow \mathscr{A}$; and $i_{\mathscr{A}}: \mathscr{A} \longrightarrow [F1, \mathscr{A}]$ is the composite of the isomorphism $\mathscr{A} \longrightarrow [1, \mathscr{A}]$ and $i_{1,\mathscr{A}}: [1, \mathscr{A}] \longrightarrow [F1, \mathscr{A}]$.

The 1-cell $e_{\mathscr{A}}: [F1, \mathscr{A}] \longrightarrow \mathscr{A}$ is strict. It is determined as such by $Ue_{\mathscr{A}}$ being the composite $T\text{-}Alg(F1, \mathscr{A}) \longrightarrow Cat(T1, A) \longrightarrow Cat(1, A) \cong A$.

The 1-cell $i_{\mathscr{A}}: \mathscr{A} \longrightarrow [F1, \mathscr{A}]$ is not strict, but rather factors through strict maps.

(We note in passing that we can describe i as the transpose of a bilinear map with underlying 1-cell

$$A \times T1 \xrightarrow{t} T(A \times 1) \xrightarrow{Tr} TA \xrightarrow{a} A$$

which is strict in $T1$ and with an A-component 2-cell which we omit.)

Note that we get $e_{\mathscr{A}}$ natural (and $i_{\mathscr{A}}$ pseudo-natural) in $\mathscr{A}$ by the corresponding properties of $e_{X,\mathscr{A}}$ and $i_{X,\mathscr{A}}$. Similarly, we see that $e_{\mathscr{A}}$ is a retract equivalence with section $i_{\mathscr{A}}$.

Composition law: Recall the multilinear map *comp*: $[\mathscr{B}, \mathscr{C}] \times [\mathscr{A}, \mathscr{B}] \to [\mathscr{A}, \mathscr{C}]$ which internalises composition. Clearly *comp* is strict in $[\mathscr{B}, \mathscr{C}]$. It transposes to $k: [\mathscr{B}, \mathscr{C}] \to [[\mathscr{A}, \mathscr{B}], [\mathscr{A}, \mathscr{C}]]$ which is a strict pseudo-map of T-algebras. The naturality of k in $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, follow from the corresponding properties for *comp*.

The axioms: We dealt with some of the axioms when constructing the data. So we already have the naturality of e and k and the retract equivalence $i \dashv e$. The final technical condition is clear as both $V(i_{\mathscr{A}} \cdot e_{\mathscr{A}})(p)$ and $e_{\mathscr{A}} \cdot [p, A] \cdot j_{\mathscr{A}}$ are the strict map $F1 \longrightarrow \mathscr{A}$ corresponding to $Up \cdot \eta: 1 \to A$. Thus we are left with the numbered axioms. The proof of these can be understood as follows. All the relevant structure maps are strict, so we need to prove equality between various parallel pairs of strict maps of T-algebras: but equality between strict maps amounts to equality of the underlying functors. Now we know that the underlying category of $[\mathscr{A}, \mathscr{B}]$ is $T\text{-}Alg(\mathscr{A}, \mathscr{B})$, the category of pseudo-maps of algebras, and we have concrete descriptions of our structure maps. So we can just check that the axioms hold at the *Cat* level.

1. By definition $j_{\mathscr{B}}: F1 \longrightarrow [\mathscr{B},\mathscr{B}]$ is the strict map corresponding under adjunction to the map $\hat{j}_{\mathscr{B}}: 1 \longrightarrow T\text{-}Alg(\mathscr{B},\mathscr{B})$ picking out the identity. Hence the composite $k \cdot j_{\mathscr{B}}: F1 \longrightarrow [[\mathscr{A},\mathscr{B}],[\mathscr{A},\mathscr{B}]]$ is strict and so uniquely determined by $\bar{e}(k \cdot j_{\mathscr{B}}) = Uk \cdot \bar{e}(j_{\mathscr{B}})$. But
$$\bar{e}(j_{\mathscr{B}}) = \hat{j}_{\mathscr{B}}: 1 \longrightarrow T\text{-}Alg(\mathscr{B},\mathscr{B})$$
picks the identity and
$$Uk: T\text{-}Alg(\mathscr{B},\mathscr{B}) \longrightarrow T\text{-}Alg([\mathscr{A},\mathscr{B}],[\mathscr{A},\mathscr{B}])$$
is functoriality data for $[\mathscr{A},-]$. So $Uk \cdot \hat{j}_{\mathscr{B}} = \hat{j}_{[\mathscr{A},\mathscr{B}]}$ by functoriality. We deduce that $k \cdot j_{\mathscr{B}} = j_{[\mathscr{A},\mathscr{B}]}$.
2. We have a composite
$$[\mathscr{A},\mathscr{C}] \longrightarrow [[\mathscr{A},\mathscr{A}],[\mathscr{A},\mathscr{C}]] \longrightarrow [F1,[\mathscr{A},\mathscr{C}]] \longrightarrow [\mathscr{A},\mathscr{C}]$$
of strict maps, which will be the identity just when it is the identity at the level of *Cat*. Applying U and factorising the middle map gives us
$$\begin{aligned} T\text{-}Alg(\mathscr{A},\mathscr{C}) &\xrightarrow{UK} T\text{-}Alg([\mathscr{A},\mathscr{A}],[\mathscr{A},\mathscr{C}]) \xrightarrow{[\varepsilon,1]} T\text{-}Alg(FU[\mathscr{A},\mathscr{A}],[\mathscr{A},\mathscr{C}]) \\ &\xrightarrow{[F\hat{j},1]} T\text{-}Alg(F1,[\mathscr{A},\mathscr{C}]) \xrightarrow{\bar{e}} Cat(1,U[\mathscr{A},\mathscr{C}]) \cong T\text{-}Alg(\mathscr{A},\mathscr{C}). \end{aligned}$$
By a series of naturalities we can chase this round to give the composite
$$\begin{aligned} Cat(1,U[\mathscr{A},\mathscr{C}]) &\xrightarrow{F} T\text{-}Alg(F1,FU[\mathscr{A},\mathscr{C}]) \\ &\xrightarrow{\bar{e}} Cat(1,UFU[\mathscr{A},\mathscr{C}]) \xrightarrow{Cat(1,\varepsilon)} Cat(1,[\mathscr{A},\mathscr{C}]), \end{aligned}$$
which is the identity by consideration of $F_s \dashv U_s$. So we are done.
3. All maps involved are strict, so we could reduce to the *Cat* level; but even there we have to prove something stronger than the plain 2-functoriality of $[\mathscr{A},-]$. It is simpler to exploit multilinearity directly. The short composite in the diagram corresponds (transposing twice) to
$$[\mathscr{C},\mathscr{D}] \cdot [\mathscr{B},\mathscr{C}] \cdot [\mathscr{A},\mathscr{B}] \xrightarrow{comp \cdot [\mathscr{A},\mathscr{B}]} [\mathscr{B},\mathscr{D}] \cdot [\mathscr{A},\mathscr{B}] \xrightarrow{comp} [\mathscr{A},\mathscr{D}],$$
where we temporarily write $\cdot$ for $\times$ to save space. On the other hand the long composite corresponds (again transposing twice) to
$$\begin{aligned} [\mathscr{C},\mathscr{D}] \cdot [\mathscr{B},\mathscr{C}] \cdot [\mathscr{A},\mathscr{B}] &\xrightarrow{1 \cdot k \cdot 1} [\mathscr{C},\mathscr{D}] \cdot [[\mathscr{A},\mathscr{B}],[\mathscr{A},\mathscr{C}]] \cdot [\mathscr{A},\mathscr{B}] \\ &\xrightarrow{1 \cdot ev} [\mathscr{C},\mathscr{D}] \cdot [\mathscr{A},\mathscr{C}] \xrightarrow{comp} [\mathscr{A},\mathscr{D}]. \end{aligned}$$
But by naturality that is equal to
$$[\mathscr{C},\mathscr{D}] \cdot [\mathscr{B},\mathscr{C}] \cdot [\mathscr{A},\mathscr{B}] \xrightarrow{[\mathscr{C},\mathscr{D}] \times comp} [\mathscr{C},\mathscr{D}] \cdot [\mathscr{A},\mathscr{C}] \xrightarrow{comp} [\mathscr{A},\mathscr{D}].$$

Thus the two sides of the diagram correspond to the two ways of associating composition to give a multilinear map $[\mathscr{C},\mathscr{D}] \times [\mathscr{B},\mathscr{C}] \times [\mathscr{A},\mathscr{B}] \to [\mathscr{A},\mathscr{D}]$. The corresponding multilinear maps $[\mathscr{C},\mathscr{D}] \times [\mathscr{B},\mathscr{C}] \times [\mathscr{A},\mathscr{B}] \times \mathscr{A} \to \mathscr{D}$ are constructed by composing three evaluations in two different orders. Since multilinear composition is associative, we are done.

4. The maps are strict so it suffices to check what happens in *Cat*. Applying U, we get on the one hand

$$T\text{-}Alg(\mathscr{A},\mathscr{B}) \xrightarrow{Uk} T\text{-}Alg([I,\mathscr{A}],[I,\mathscr{B}]) \xrightarrow{T\text{-}Alg(1,e)} T\text{-}Alg([I,\mathscr{A}],\mathscr{B})$$

and on the other

$$T\text{-}Alg(\mathscr{A},\mathscr{B}) \xrightarrow{T\text{-}Alg(e,1)} T\text{-}Alg([I,\mathscr{A}],\mathscr{B}).$$

The equality of these just expresses naturality of e, so we are done.

5. Trivial as we defined $j_{\mathscr{A}}$ so that $Ue_{[\mathscr{A},\mathscr{A}]}(j_{\mathscr{A}}) = \mathrm{id}_{\mathscr{A}}$.

This gives us the main result at which we have been aiming.

Proposition 22. *If T is a pseudo-commutative 2-monad on Cat, then T-Alg is a pseudo-closed 2-category.*

We can say something more about the biadjunction $F \dashv U$. Recall the notion of closed functor which we introduced in Section 2. For U, we have obvious data

$$\psi: 1 \xrightarrow{\eta_1} UF1, \quad \psi: U[\mathscr{A},\mathscr{B}] = T\text{-}Alg(\mathscr{A},\mathscr{B}) \xrightarrow{U} Cat(A,B) = [U\mathscr{A},U\mathscr{B}]$$

and it is easy to check that this makes U a closed functor. For F, we have data

$$\phi: F1 \xrightarrow{\mathrm{id}} F1, \quad \phi: F[X,Y] \to [FX,FY],$$

where the latter is the strict map of T-algebras corresponding to the action $F:[X,Y] \to T\text{-}Alg(FX,FY) = U[FX,FY]$. (Equally it is the transpose of F applied to evaluation in *Cat*.) Again it is easy to check that this data makes U a closed functor. Finally, one can extend the definition of Eilenberg and Kelly [7] to define the notion of a closed pseudo-natural transformation between closed functors. (We omit the details.) We then observe that $\eta: 1_{Cat} \to UF$ is a closed natural transformation, while $\varepsilon: FU \to 1_{T\text{-}Alg}$ is a closed pseudo-natural transformation. Thus we state our main theorem as follows.

Theorem 12. *If T is a pseudo-commutative 2-monad on Cat, then*

(i) *T-Alg is a pseudo-closed 2-category,*
(ii) *U and F are closed 2-functors, and*
(iii) *$F \dashv U$ is a closed biadjunction.*

6.5. Symmetric structure

Everything we did in the previous section went through for a general pseudo-commutativity γ on T. Now let us assume that γ is symmetric so that *T-Alg* is a symmetric 2-multicategory. Then we can define a symmetry for T-Alg as a pseudo-closed 2-category.

Symmetry: We have an isomorphism $\tilde{\tau}$ say, given by the following composite:

$$\begin{aligned} T\text{-}Alg([\mathscr{A},[\mathscr{B},\mathscr{C}]],[\mathscr{A},[\mathscr{B},\mathscr{C}]]) &\cong T\text{-}Alg([\mathscr{A},[\mathscr{B},\mathscr{C}]]\times\mathscr{A}\times\mathscr{B};\mathscr{C}) \\ &\cong T\text{-}Alg([\mathscr{A},[\mathscr{B},\mathscr{C}]]\times\mathscr{B}\times\mathscr{A};\mathscr{C}) \\ &\cong T\text{-}Alg([\mathscr{A},[\mathscr{B},\mathscr{C}]],[\mathscr{B},[\mathscr{A},\mathscr{C}]]). \end{aligned}$$

Then we set $\tau = U\tilde{\tau}(\mathrm{id})$. Equivalently,

$$U\tau : T\text{-}Alg(\mathscr{A},[\mathscr{B},\mathscr{C}]) \to T\text{-}Alg(\mathscr{B},[\mathscr{A},\mathscr{C}])$$

corresponds under exponential transpose to the symmetry

$$T\text{-}Alg(\mathscr{A},\mathscr{B};\mathscr{C}) \to T\text{-}Alg(\mathscr{B},\mathscr{A};\mathscr{C}).$$

Equivalently composition with $\tau : [\mathscr{A},[\mathscr{B},\mathscr{C}]] \to [\mathscr{B},[\mathscr{A},\mathscr{C}]]$ induces maps

$$T\text{-}Alg(\mathscr{D},[\mathscr{A},[\mathscr{B},\mathscr{C}]]) \to T\text{-}Alg(\mathscr{D},[\mathscr{B},[\mathscr{A},\mathscr{C}]])$$

which correspond under exponential transpose to the symmetry

$$T\text{-}Alg(\mathscr{D},\mathscr{A},\mathscr{B};\mathscr{C}) \to T\text{-}Alg(\mathscr{D},\mathscr{B},\mathscr{A};\mathscr{C}).$$

Now we check the axioms we have given for a symmetry.

- *Identity law for e*. Since e and τ are strict we can check this at the *Cat* level. We recall that $\bar{e}_{X,\mathscr{D}} : T\text{-}Alg(FX,\mathscr{D}) \to T\text{-}Alg(X,\mathscr{D}) = Cat(X,D)$ is induced by composition with the trivial parametrised map $X \to FX$. Applying this in case $\mathscr{D} = [\mathscr{A},\mathscr{C}]$, we find that $Ue_{[\mathscr{A},\mathscr{C}]}$ corresponds to

  $$T\text{-}Alg(F1,\mathscr{A};\mathscr{C}) \to T\text{-}Alg(1,\mathscr{A};\mathscr{C}) \cong T\text{-}Alg(\mathscr{A},\mathscr{C})$$

  induced by composition with $1 \to F1$. Since $U\tau$ corresponds to the symmetry $T\text{-}Alg(\mathscr{A},F1;\mathscr{C}) \to T\text{-}Alg(F1,\mathscr{A};\mathscr{C})$, the composite $Ue_{[\mathscr{A},\mathscr{C}]} \cdot Uc$ corresponds to

  $$T\text{-}Alg(\mathscr{A},F1;\mathscr{C}) \to T\text{-}Alg(\mathscr{A},1;\mathscr{C}) \cong T\text{-}Alg(\mathscr{A},\mathscr{C})$$

  induced by composition with $1 \to F1$. But, as we saw in 6.3, that is exactly what $U[\mathscr{A},e_{\mathscr{C}}]$ corresponds to.
- *Identity law for i*. Recall that $i_{X,\mathscr{B}} : [X,\mathscr{B}] \to [FX,\mathscr{B}]$ corresponds to the multilinear map $[X,\mathscr{B}]\times FX \to \mathscr{B}$ strict in FX which corresponds to the canonical parametrised map $[X,\mathscr{B}]\times X \to \mathscr{B}$. Hence $i_{[\mathscr{A},\mathscr{C}]}$ corresponds to the trivially parametrised identity $[\mathscr{A},\mathscr{C}]\times 1 \to [\mathscr{A},\mathscr{C}]$, so to the trivially parametrised evaluation $[\mathscr{A},\mathscr{C}]\times 1\times\mathscr{A} \to \mathscr{C}$. Thus the composite $\tau \cdot i_{[\mathscr{A},\mathscr{C}]}$ corresponds to the trivially parametrised evaluation $[\mathscr{A},\mathscr{C}]\times\mathscr{A}\times 1 \to \mathscr{C}$. But since the partial forgetful operations respect composition, that is exactly what $[\mathscr{A},i_{\mathscr{C}}]$ corresponds to.

- *Yang–Baxter Law*. By naturality considerations this is immediate from the symmetry of the 2-multicategory *T-Alg*.

We deduce a result which applies to the cases of greatest interest to us.

Theorem 13. *If T is a symmetric pseudo-commutative 2-monad on Cat, then*

(i) *T-Alg is a symmetric pseudo-closed 2-category,*
(ii) *U and F are closed 2-functors, and*
(iii) *$F \dashv U$ is a closed biadjunction.*

6.6. Pseudo-monoidal structure

Now for simplicity of exposition suppose that our 2-monad T on *Cat* is finitary. (We recall in passing that a finitary monad on *Cat* has at most one enrichment in *Cat*, so that the enrichment T corresponding to the strength t is determined.) We wish to check that under these circumstances we have for each $\mathscr{A}$ a biadjoint to $[\mathscr{A}, -] : T\text{-}Alg \to T\text{-}Alg$. In view of Theorem 2 this will allow us to deduce that $T\text{-}Alg$ is a pseudo-monoidal pseudo-closed 2-category. The existence of a biadjoint follows from [3] together with some observations which we sketch here.

First observe that for every T-algebra $\mathscr{A}$, the 2-functor

$$T\text{-}Alg_s \xrightarrow{J} T\text{-}Alg \xrightarrow{T\text{-}Alg(\mathscr{A},-)} Cat$$

preserves limits, as J has a left 2-adjoint and because representables always preserve limits. As T is finitary, the 2-category $T\text{-}Alg_s$ is locally finitely presentable, so the 2-functor $T\text{-}Alg(\mathscr{A},-)\cdot J$ has a left 2-adjoint. We want more. $T\text{-}Alg(\mathscr{A},-)\cdot J$ is equal to the composite of $[\mathscr{A},-] : T\text{-}Alg_s \to T\text{-}Alg_s$, with the forgetful $U_s : T\text{-}Alg_s \to Cat$. Since U_s creates limits, it follows that in fact $[\mathscr{A},-] : T\text{-}Alg_s \to T\text{-}Alg_s$ preserves limits. So it also has a left adjoint $-\oslash\mathscr{A}$.

We can describe the left adjoint concretely as follows. We have

$$T\text{-}Alg_s(A \oslash \mathscr{B}, \mathscr{C}) \cong T\text{-}Alg_s(\mathscr{A}, [\mathscr{B}, \mathscr{C}]).$$

Now the right-hand side is isomorphic to the full subcategory of the category $T\text{-}Alg(\mathscr{A}, \mathscr{B}; \mathscr{C})$ on those multilinear maps which are strict in $\mathscr{A}$. So we seek a representation of the 2-functor $T\text{-}Alg_s \to Cat$. Since $T\text{-}Alg_s$ is complete, we can construct this as a colimit: we take a 1-cell $p : F(A \times B) \to \mathscr{A} \oslash \mathscr{B}$ together with a 2-cell

$$\begin{tikzcd}
F(T(A\times B)) \arrow[r, "\mu"] & F(A\times B) \arrow[dr, "p"] & \\
F(A\times TB) \arrow[u, "Ft"] \arrow[r, "F(A\times b)"'] \arrow[ur, phantom, "\Downarrow\alpha"] & F(A\times B) \arrow[r, "p"'] & \mathscr{A}\oslash\mathscr{B}
\end{tikzcd}$$

(Here we write μ also for the standard strict map between free T-algebras given by μ.) This data should be universal in $T\text{-}Alg_s$ with the following properties.

1. The diagram

$$\begin{tikzcd}
F(T(A\times B)) \arrow[r, "\mu"] & F(A\times B) \arrow[dr, "p"] & \\
F(TA\times B) \arrow[u, "Ft^*"] \arrow[r, "F(A\times b)"'] & F(A\times B) \arrow[r, "p"'] & \mathscr{A}\oslash\mathscr{B}
\end{tikzcd}$$

(dual to that for the 2-cell α) commutes.
2. $\alpha\cdot F(A\times\eta)$ is an identity 2-cell.
3. $\alpha\cdot F(A\times\mu)$ is equal to the pasting

$$\begin{tikzcd}
FT^2(A\times B) \arrow[r, "F\mu"] \arrow[dr, phantom, "\Downarrow FU\alpha"] & FT(A\times B) \arrow[r, "FUp"] & FU(\mathscr{A}\oslash\mathscr{B}) \arrow[dr] & & \\
FT(A\times TB) \arrow[u, "FTt"] \arrow[r, "FT(A\times b)"'] & FT(A\times B) \arrow[ur, "FUp"'] \arrow[r, "\mu"] \arrow[dr, phantom, "\Downarrow\alpha"] & F(A\times B) \arrow[r, "p"] & \mathscr{A}\oslash\mathscr{B} \\
F(A\times T^2B) \arrow[u, "Ft"] \arrow[r, "F(A\times Tb)"'] & F(A\times TB) \arrow[u, "Ft"] \arrow[r, "F(A\times b)"'] & F(A\times B) \arrow[ur, "p"'] &
\end{tikzcd}$$

(Here the unlabelled arrow is given by the structure map for the T-algebra $\mathscr{A}\oslash\mathscr{B}$.) We can construct this universal object using a coequalizer, an iso-coinserter and two coequifiers in $T\text{-}Alg_s$.

With the operation $\oslash$ in place we state our final result.

Theorem 14. *Let T be a finitary pseudo-commutative 2-monad on Cat. Then the 2-category T-Alg has a pseudo-monoidal pseudo-closed structure induced by its pseudo-closed structure. Furthermore, U is a pseudo-monoidal functor and the left biadjoint F a strong pseudo-monoidal functor.*

Proof. We have a natural isomorphism

$$T\text{-}Alg_s(\mathscr{A}\oslash\mathscr{B},\mathscr{C})\cong T\text{-}Alg_s(\mathscr{A},[\mathscr{B},\mathscr{C}]).$$

in $T\text{-}Alg_s$. The adjunction $(-)'\dashv J$ from [3] gives retract equivalences in $T\text{-}Alg$ so in that 2-category we get a diagram natural in $\mathscr{C}$

$$d_{\mathscr{C}}=[\mathscr{A}\oslash\mathscr{B},\mathscr{C}]\xrightarrow{k_{\mathscr{B}}}[[\mathscr{B},\mathscr{A}\oslash\mathscr{B}],[\mathscr{B},\mathscr{C}]]\xrightarrow{[\mathrm{unit},[\mathscr{B},\mathscr{C}]]}[\mathscr{A},[\mathscr{B},\mathscr{C}]]$$

using the unit from the natural isomorphism. The diagram of Theorem 2 then commutes because it does so in Cat and the maps concerned are strict. Now we know, by a general result in [3], that Ud presents $(-\oslash\mathscr{C}):T\text{-}Alg\to T\text{-}Alg$ as a left biadjoint to $[\mathscr{B},-]:T\text{-}Alg\to T\text{-}Alg$; it follows that d is an equivalence. So by Theorem 2 we get the pseudo-monoidal structure. The claims concerning U and F follow from corresponding properties of U and F as closed functors. $\square$

As it stands our result relies on calculations which we have indicated but have not given in Section 2. As an alternative one could use the concrete description of $\mathscr{A} \oslash \mathscr{B}$ above to establish the pseudo-monoidal structure directly. Even then there is much to check.

Acknowledgements

We are grateful to a referee for some perceptive remarks and suggestions concerning the material of this paper.

References

[1] M. Barr, C. Wells, Toposes, Triples, and Theories, Grundlehren der Mathematischen Wissenschaften, Vol. 278, Springer, Berlin, 1985.
[2] G.J. Bird, G.M. Kelly, A.J. Power, R. Street, Flexible limits for 2-categories, J. Pure Appl. Algebra 61 (1989) 1–27.
[3] R. Blackwell, G.M. Kelly, A.J. Power, Two-dimensional monad theory, J. Pure Appl. Algebra 59 (1989) 1–41.
[4] B. Day, On closed categories of functors, Reports of the Midwest Category Seminar IV, Lecture Notes in Mathematics, Vol. 137, Springer, Berlin, 1970, pp. 1–38.
[5] B. Day, On closed categories of functors II, Category Seminar, Lecture Notes in Mathematics, Vol. 420, Springer, Berlin, 1974, pp. 20–54.
[6] B. Day, An embedding theorem for closed categories, Category Seminar, Lecture Notes in Mathematics, Vol. 420, Springer, Berlin, 1974, pp. 55–64.
[7] S. Eilenberg, G.M. Kelly, Closed categories, Proceedings of the Conference on Categorical Algebra, La Jolla, 1965, Springer, Berlin, 1966.
[8] R. Gordon, A.J. Power, R. Street, Coherence for tricategories, Mem. Amer. Math. Soc. 558 (1995).
[9] M. Hyland, A.J. Power, Symmetric monoidal sketches, Proceedings of the Second Conference on Principles and Practice of Declarative Programming, 2000, pp. 280–288.
[10] G.M. Kelly, Coherence theorems for lax algebras and for distributive laws, Category Seminar, Lecture Notes in Mathematics, Vol. 420, Springer, Berlin, 1974, pp. 281–375.
[11] A. Kock, Monads on symmetric monoidal closed categories, Arch. Math. 21 (1970) 1–10.
[12] A. Kock, Closed categories generated by commutative monads, J. Austral. Math. Soc. 12 (1971) 405–424.
[13] A. Kock, Strong Functors and Monoidal Monads, Arch. Math. 23 (1972) 113–120.
[14] M. Laplaza, Embedding of closed categories into monoidal closed categories, Trans. Amer. Math. Soc. 233 (1977) 85–91.
[15] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics, Vol. 5, Springer, Berlin, 1971.
[16] A.J. Power, E.P. Robinson, A characterization of PIE limits, Math. Proc. Cambridge Philos. Soc. 110 (1991) 33–47.
[17] R.H. Street, The formal theory of monads, J. Pure Appl. Algebra 2 (1972) 149–168.